JVR

Journal of Vectorial Relativity
JVR **4** (2009) **3** 43-48

Orthogonal Transformations via Quaternions

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***ABSTRACT**.-* We study a triple quaternionic product to generate 4x4 orthogonal matrices, which leads in natural manner to Dirac matrices and rotations in three and four dimensions.

***KEY WORDS*:** Quaternions; Dirac matrices; spatial rotations; Lorentz Transformations

I. INTRODUCTION

The quaternionic units obey the algebra [1-6]:

$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = -1, \quad \mathbf{IJK} = -1, \tag{1}$$

then it is possible to realize the product:

$$\tilde{\mathbf{F}} = \mathbf{pF}, \tag{2.a}$$

with

$$\mathbf{F} = F_1\mathbf{I} + F_2\mathbf{J} + F_3\mathbf{K} + F_4, \tag{2.b}$$

$$\mathbf{p} = p_1\mathbf{I} + p_2\mathbf{J} + p_3\mathbf{K} + p_4, \tag{2.c}$$

thus $\tilde{\mathbf{F}}$ can be written in the matrix form:

$$\begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \\ \tilde{F}_4 \end{pmatrix} = \underbrace{\begin{pmatrix} p_4 & -p_3 & p_2 & p_1 \\ p_3 & p_4 & -p_1 & p_2 \\ -p_2 & p_1 & p_4 & p_3 \\ -p_1 & -p_2 & -p_3 & p_4 \end{pmatrix}}_{\underset{\sim}{P}} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}. \tag{3}$$

The magnitude of $\mathbf{F}$ is defined by:

$$|\mathbf{F}|^2 = \mathbf{F}\overline{\mathbf{F}} = \overline{\mathbf{F}}\mathbf{F} = F_1^2 + F_2^2 + F_3^2 + F_4^2, \tag{4.a}$$

with

$$\overline{\mathbf{F}} = -F_1\mathbf{I} - F_2\mathbf{J} - F_3\mathbf{K} + F_4, \tag{4.b}$$

and we note that for any $\mathbf{A}$ and $\mathbf{B}$:

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$$\overline{\mathbf{AB}} = \overline{\mathbf{B}}\,\overline{\mathbf{A}}\ . \tag{4.c}$$

Then from (2.a, 3, 4.a,c):

$$\left|\tilde{\mathbf{F}}\right|^2 = \mathbf{p}\left|\mathbf{F}\right|^2\overline{\mathbf{p}} = \mathbf{p}\overline{\mathbf{p}}\left|\mathbf{F}\right|^2, \quad \det \underset{\sim}{\mathrm{P}} = \left(\mathbf{p}\overline{\mathbf{p}}\right)^2, \tag{5.a}$$

and if $\mathbf{p}$ is unitary:

$$\mathbf{p}\overline{\mathbf{p}} = p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1, \quad \det \underset{\sim}{\mathrm{P}} = 1, \tag{5.b}$$

the magnitude of $\mathbf{F}$ is conserved:

$$\tilde{F}_1^2 + \tilde{F}_2^2 + \tilde{F}_3^2 + \tilde{F}_4^2 = F_1^2 + F_2^2 + F_3^2 + F_4^2, \tag{6}$$

that is:

$$\begin{pmatrix} \tilde{F}_1 & \tilde{F}_2 & \tilde{F}_3 & \tilde{F}_4 \end{pmatrix}\begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \\ \tilde{F}_4 \end{pmatrix} = \begin{pmatrix} F_1 & F_2 & F_3 & F_4 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix},$$

which implies that (3) is an orthogonal transformation :

$$\underset{\sim}{\mathrm{P}}^{\mathrm{T}}\mathrm{P} = \underset{\sim}{\mathrm{I}}_{4\times 4}\,. \tag{7}$$

A matrix is orthogonal if the multiplication with its transpose gives the unit matrix, and its principal property is the conservation of the magnitude of vectors under its action.

Similarly, the product:

$$\tilde{\mathbf{F}} = \mathbf{F}\mathbf{q}\,, \tag{8.a}$$

has the matrix representation:

$$\begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \\ \tilde{F}_4 \end{pmatrix} = \underbrace{\begin{pmatrix} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{pmatrix}}_{\underset{\sim}{\mathrm{Q}}}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}, \tag{8.b}$$

and if $\mathbf{q}$ is unitary then (6) is verified and therefore $\underset{\sim}{\mathrm{Q}}$ is orthogonal with $\det \underset{\sim}{\mathrm{Q}} = 1$.

The cases (2.a, 8.a) can be unified in one scheme employing the triple quaternionic product:

$$\tilde{\mathbf{F}} = \mathbf{p}\mathbf{F}\mathbf{q}\,, \tag{9.a}$$

that is:

$$\begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \\ \tilde{F}_4 \end{pmatrix} = \underset{\sim}{\mathrm{D}}_{4\times4} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}, \quad \underset{\sim}{\mathrm{D}} = \underset{\sim}{\mathrm{P}}\, \underset{\sim}{\mathrm{Q}}, \tag{9.b}$$

then $\left|\tilde{\mathbf{F}}\right|^2 = \left(\mathbf{p}\overline{\mathbf{p}}\right)\left(\mathbf{q}\overline{\mathbf{q}}\right)\left|\mathbf{F}\right|^2$, and thus it is clear that (6) is satisfied for the unitary quaternions $\mathbf{p}$ and $\mathbf{q}$, with the orthogonal matrices $\underset{\sim}{\mathrm{P}}$ y $\underset{\sim}{\mathrm{Q}}$, implying that $\underset{\sim}{\mathrm{D}}$ also is an orthogonal matrix:

$$\underset{\sim}{\mathrm{D}}^{\mathrm{T}} \underset{\sim}{\mathrm{D}} = \underset{\sim}{\mathrm{I}}, \quad \det \underset{\sim}{\mathrm{D}} = 1. \tag{9.c}$$

In the following pages we realize applications of (9.a.b.c): Sec. 2 shows that $\underset{\sim}{\mathrm{D}}$ reproduces the 16 Dirac matrices [7] if $\mathbf{p}$ and $\mathbf{q}$ coincide with quaternionic units. Sec. 3 considers the case of special relativity because $\underset{\sim}{\mathrm{D}}$ generates Lorentz transformations when $\mathbf{q} = \overline{\mathbf{p}}^*$. Sec. 4 is dedicated to 3-rotations because now we ask that the unitary quaternion $\mathbf{p}$ to be real ($\mathbf{p} = \overline{\mathbf{p}}^*$), and therefore $\mathbf{q} = \overline{\mathbf{p}}$.

II. DIRAC MATRICES

In relativistic quantum mechanics are important the 16 Dirac matrices [7]:

$$\underset{\sim}{\mathrm{I}}, \quad \gamma^0 = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & \text{-I} \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix}, \quad \gamma^0\gamma^5 = \begin{pmatrix} 0 & \mathrm{I} \\ \text{-I} & 0 \end{pmatrix},$$

$$\gamma^r = \begin{pmatrix} 0 & \sigma_r \\ -\sigma_r & 0 \end{pmatrix}, \quad \gamma^r\gamma^5 = \begin{pmatrix} \sigma_r & 0 \\ 0 & -\sigma_r \end{pmatrix}, \quad \sigma^{0r} = -\sigma^{r0} = i\begin{pmatrix} 0 & \sigma_r \\ \sigma_r & 0 \end{pmatrix}, \quad r = 1,2,3, \tag{10}$$

$\sigma^{jk} = -\sigma^{kj} = \begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix}$, (jkl) is a cyclic permutation of (123),

and σ_j are the Pauli matrices [7,8]:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i = \sqrt{-1}. \tag{11}$$

Now in (9.a) we may select to $\mathbf{p}$ y $\mathbf{q}$ as the quaternionic units, for example, if $\mathbf{p} = 1$, $\mathbf{q} = \mathbf{K}$, then from (9.b) we obtain that $\underset{\sim}{\mathrm{D}} = i\sigma^{31}$. In similar form, if $\mathbf{p} = \mathbf{I}$, $\mathbf{q} = \mathbf{J}$ we deduce that $\underset{\sim}{\mathrm{D}} = -\gamma^1\gamma^5$, etc. Thus it results the Table:

$\mathbf{p} \backslash \mathbf{q}$	1	$\mathbf{I}$	$\mathbf{J}$	$\mathbf{K}$
1	$\underset{\sim}{\mathrm{I}}$	γ^1	$-\gamma^3$	$i\sigma^{31}$
$\mathbf{I}$	σ^{02}	$-\gamma^3\gamma^5$	$-\gamma^1\gamma^5$	$-\gamma^5$
$\mathbf{J}$	$\gamma^0\gamma^5$	σ^{32}	σ^{12}	$i\gamma^2$
$\mathbf{K}$	$-i\gamma^2\gamma^5$	$i\sigma^{03}$	$i\sigma^{01}$	γ^0

(12)

If $\underset{\sim}{\mathrm{W}}$ denotes to any matrix from this Table, then it is easy to prove that:

$$\underset{\sim}{\mathrm{W}}^{-1}=\underset{\sim}{\mathrm{W}}^{\mathrm{T}}, \quad \underset{\sim}{\mathrm{W}}^{*}=\underset{\sim}{\mathrm{W}}, \quad \underset{\sim}{\mathrm{W}}^{\mathrm{T}}=\pm\underset{\sim}{\mathrm{W}}, \tag{13}$$

that is, are orthogonal, real and symmetric or antisymmetric matrices.

III. LORENTZ TRANSFORMATIONS

In Minkowski space [9] the Lorentz transformations connect, linearly, the coordinates of an event from two frames in uniform relative motion (*c* is the light velocity in vacuum):

$$\begin{pmatrix} i\tilde{x} \\ i\tilde{y} \\ i\tilde{z} \\ c\tilde{t} \end{pmatrix} = \underset{\sim}{\mathrm{D}} \begin{pmatrix} ix \\ iy \\ iz \\ ct \end{pmatrix}, \tag{14.a}$$

that in geometrical terms corresponds to a 4-rotation preserving magnitudes:

$$\left(c\tilde{t}\right)^2 - \tilde{x}^2 - \tilde{y}^2 - \tilde{z}^2 = \left(ct\right)^2 - x^2 - y^2 - z^2, \tag{14.b}$$

which is a requisite from special relativity postulates.

If the quaternion (2.b) is selected as:

$$\mathbf{F} = ix\mathbf{I} + iy\mathbf{J} + iz\mathbf{K} + ct, \tag{15}$$

then (6) and (9.b) reproduce (14.b) and (14.a), respectively, and (9.a) gives:

$$i\tilde{x}\mathbf{I} + i\tilde{y}\mathbf{J} + i\tilde{z}\mathbf{K} + c\tilde{t} = \mathbf{p}\left(ix\mathbf{I} + iy\mathbf{J} + iz\mathbf{K} + ct\right)\mathbf{q} \tag{16.a}$$

and after application of the operation $*$:

$$-i\tilde{x}\mathbf{I} - i\tilde{y}\mathbf{J} - i\tilde{z}\mathbf{K} + c\tilde{t} = \mathbf{p}^*\left(-ix\mathbf{I} - iy\mathbf{J} - iz\mathbf{K} + ct\right)\mathbf{q}^*,$$

which under the operation – implies [to remember (4.b,c)]:

$$i\tilde{x}\mathbf{I} + i\tilde{y}\mathbf{J} + i\tilde{z}\mathbf{K} + c\tilde{t} = \overline{\mathbf{q}}^*\left(ix\mathbf{I} + iy\mathbf{J} + iz\mathbf{K} + ct\right)\overline{\mathbf{p}}^* \tag{16.b}$$

thus the identity between (16.a) and (16.b) is obtained if $\overline{\mathbf{q}}^* = \mathbf{p}$, that is:

$$\mathbf{q} = \overline{\mathbf{p}}^{\,*}, \tag{17}$$

being $\mathbf{p}$ unitary. Therefore, (9.a) adopts the structure [9-13]:

$$\tilde{\mathbf{F}} = \mathbf{p}\mathbf{F}\overline{\mathbf{p}}^{*}, \quad \mathbf{p}\overline{\mathbf{p}} = 1, \tag{18}$$

that with (15) permit to generate homogeneous Lorentz transformations verifying (9.c), only it is necessary to give $\mathbf{p}$. In fact, with (2.c, 14.a, 15, 18) we deduce the expressions for $\underset{\sim}{D}$ from the literature [4, 12-15]. For example, if we choose:

$$\mathbf{p} = -i\, senh\left(\frac{\tau}{2}\right)\mathbf{K} + cosh\left(\frac{\tau}{2}\right), \ \tau \text{ real}, \tag{19.a}$$

we obtain the known Lorentz formulae:

$$\begin{gathered} \tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{z} = \gamma(z - vt), \quad \tilde{t} = \gamma\left(t - \frac{v}{c^2} z\right) \\ \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad tanh(\tau) = \frac{v}{c} < 1 \end{gathered} \tag{19.b}$$

for two observers in the z direction with relative velocity v.

IV. SPATIAL ROTATIONS

Here we consider Lorentz transformations without changes in the temporal coordinate:

$$\tilde{t} = t, \tag{20}$$

which corresponds to 3-rotations. If we employ (20) in (16.a) then it is clear that t can be eliminated identically, preserving only spatial variables, if $\mathbf{pq} = 1$, which from (17) gives $\mathbf{p}\overline{\mathbf{p}}^{*} = 1$, equivalent to $\mathbf{p}^{*}\overline{\mathbf{p}} = 1$, thus (5.b) imposes the condition:

$$\mathbf{p} = \mathbf{p}^{*} \tag{21}$$

that is, the Lorentz matrices generate 3-rotations when in (9.a) $\mathbf{p}$ is unitary and real. Then (14.a) admits the expression:

$$\begin{pmatrix} i\tilde{x} \\ i\tilde{y} \\ i\tilde{z} \\ c\tilde{t} \end{pmatrix} = \begin{pmatrix} & & & 0 \\ & \underset{\sim}{R}_{3\times 3} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ix \\ iy \\ iz \\ ct \end{pmatrix},$$

implying (20) and:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \underset{\sim}{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \underset{\sim}{R}\underset{\sim}{R}^{T} = \underset{\sim}{I}, \quad \det \underset{\sim}{R} = 1, \qquad (22.a)$$

respecting the invariance (14.b) in its pythagoric form $\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = x^2 + y^2 + z^2$.

With (9.a, 17, 21) it is immediate to obtain the structure of $\underset{\sim}{R}$ reported in the literature [13]:

$$\underset{\sim}{R} = \begin{pmatrix} 1-2\left(p_2^2 + p_3^2\right) & 2\left(p_1 p_2 - p_3 p_4\right) & 2\left(p_1 p_3 + p_2 p_4\right) \\ 2\left(p_1 p_2 + p_3 p_4\right) & 1-2\left(p_1^2 + p_3^2\right) & 2\left(p_2 p_3 - p_1 p_4\right) \\ 2\left(p_1 p_3 - p_2 p_4\right) & 2\left(p_1 p_4 + p_2 p_3\right) & 1-2\left(p_1^2 + p_2^2\right) \end{pmatrix}, \qquad (22.b)$$

thus (22.a, b) give us a systematic process to produce spatial rotations, only it is necessary to select the four real values p_j verifying (5.b).

REFERENCES

[1] J. Kronsbein , "*Kinematics-quaternions-spinors-and Pauli's spin matrices*", Am. J. Phys. 35, No.4 (1967) 335-342.

[2] B. L. van der Waerden, "*Hamilton's discovery of quaternions*", Math. Mag. 49, No.5 (1976) 227-234.

[3] L. Byron McAllister, "*A quick introduction to quaternions*", Pi Mu Epsilon J. 9, No.1 (1989) 23-25.

[4] R. Penrose, "*The road to reality*", Jonathan Cape, London (2004).

[5] N. Hamdan, I. Guerrero M., J. López-Bonilla and L. Rosales R., "*On the Faraday's complex vector*", The Icfai Univ. J. Phys. 1, No.3 (2008) 52-56.

[6] G. Martínez-Sierra and P. F. Benoit-Poirier, "*Una epistemología histórica del producto vectorial: Del cuaternión al análisis vectorial*", Lat. Am. J. Phys. Educ. 2, No.2 (2008) 201-208.

[7] J. Leite-Lopes, "*Introduction to quantum electrodynamics*", Ed. Trillas, Mexico (1970).

[8] W. Pauli, "*Uber das wasserstoffspektrum vom standpunkt der neuen quantenmechanik*", Zeits. für Physik 37 (1926) 263-277.

[9] J. L. Synge, "*Relativity: the special theory*", North-Holland Pub., Amsterdan (1965).

[10] C. Lanczos, "*The variational principles of mechanics*", University of Toronto Press (1970).

[11] S. De Leo, "*Quaternions and special relativity*", J. Math. Phys. 37, No.6 (1996) 2955-2968.

[12] M. Acevedo, J. López-Bonilla and M. Sánchez, "*Quaternions, Maxwell equations and Lorentz transformations*", Apeiron 12, No.4 (2005) 371-384.

[13] I. Guerrero M., J. López-Bonilla and L. Rosales R., "*Rotations in three and four dimensions via 2x2 complex matrices and quaternions*", The Icfai Univ. J. Phys. 1, No.2 (2008) 7-13.

[14] J. López-Bonilla, J. Morales and G. Ovando, "*On the homogeneous Lorentz transformation*", Bull. Allahabad Math. Soc. 17 (2002) 53-58.

[15] B.E. Carvajal-Gámez, M. Galaz L. and J. López-Bonilla, "*On the Lorentz matrix in terms of Infeld-van der Waerden symbols*", Scientia Magna 3, No.3 (2007) 56-57.