

# Faraday Tensor: its Algebraic Structure

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**ABSTRACT.** We study the eigenvalue problem of Faraday tensor.

**KEYWORDS:** Maxwell equations; Faraday tensor.

## I. INTRODUCTION

The Maxwell equations in vacuum ( $c$  means the light velocity):

$$\vec{\nabla} \cdot \vec{B} = 0 \quad , \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad , \quad (1.a)$$

$$\vec{\nabla} \cdot \vec{E} = 0 \quad , \quad \vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \quad , \quad (1.b)$$

adopt the tensorial form [1,2]:

$$F_{ab,j} + F_{ja,b} + F_{bj,a} = 0 \quad , \quad (2.a)$$

$$F^a_{b,a} = 0 \quad , \quad (2.b)$$

where we accept the Einstein's convention of sum on a repeated index, with:

$$(x^j) = (ct, x, y, z) \quad \mathcal{L} \quad , \quad r = \frac{\partial}{\partial x^r} \quad , \quad (3)$$

The Minkowski metric is  $(1, -1, -1, -1)$  and the Faraday matrix are given in their different representations:

$$(F^{ab}) = \begin{pmatrix} 0 & E_X & E_Y & E_Z \\ -E_X & 0 & cB_Z & -cB_Y \\ -E_Y & -cB_Z & 0 & cB_X \\ -E_Z & cB_Y & -cB_X & 0 \end{pmatrix} \quad , \quad (F_{ab}) = \begin{pmatrix} 0 & -E_X & -E_Y & -E_Z \\ E_X & 0 & cB_Z & -cB_Y \\ E_Y & -cB_Z & 0 & cB_X \\ E_Z & cB_Y & -cB_X & 0 \end{pmatrix} \quad , \quad (4.a)$$

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$$\tilde{F} \equiv (F^a_b) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix} \quad (4.b)$$

The expressions (2.a) and (2.b) reproduce to (1.a) and (1.b), respectively.

It is clear that if in (1.a) we make the changes:

$$\vec{E} \rightarrow c\vec{B}, \quad c\vec{B} \rightarrow -\vec{E}, \quad (5)$$

then results (1.b); and with (5) into (1.b) we obtain (1.a). This motivates the introduction of the Faraday Dual matrix :

$$(*F^{ab}) = \begin{pmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & -E_z & E_y \\ -cB_y & E_z & 0 & -E_x \\ -cB_z & -E_y & E_x & 0 \end{pmatrix}, \quad (*F_{ab}) = \begin{pmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & -E_z & E_y \\ cB_y & E_z & 0 & -E_x \\ cB_z & -E_y & E_x & 0 \end{pmatrix}, \quad (6.a)$$

$$*\tilde{F} \equiv (*F^a_b) = \begin{pmatrix} 0 & -cB_x & -cB_y & -cB_z \\ -cB_x & 0 & -E_z & -E_y \\ -cB_y & -E_z & 0 & -cB_x \\ -cB_z & E_y & -E_x & 0 \end{pmatrix} \quad (6.b)$$

constructed from (4) and (5), which permits to write the Maxwell equations (2) in the form:

$$*F^a_{b,a} = 0, \quad (7.a)$$

$$*F_{ab,j} + *F_{ja,b} + *F_{bj,a} = 0 \quad (7.b)$$

The Levi–Civita antisymmetric tensor:

$$\eta^{abpq} = \epsilon^{abpq}, \quad \eta_{abpq} = -\epsilon_{abpq} \quad (8.a)$$

with the symbols:

$$\epsilon^{abpq} = \epsilon_{abpq} = \begin{cases} 1, & \text{if } (abpq) \text{ is even permutation of } (0123), \\ -1, & \text{if } (abpq) \text{ is odd permutation of } (0123), \\ 0, & \text{otherwise,} \end{cases} \quad (8.b)$$

permits to establish a relationship between Faraday tensor and its Dual:

$$*F^{ab} = \frac{1}{2} \eta^{abpq} F_{pq} \quad , \quad (9.a)$$

or equivalently

$$F^{ab} = -\frac{1}{2} \eta^{abpq} *F_{pq} \quad , \quad (9.b)$$

which shows that (6) may be constructed via (4,9.a), that is, (9.a) is the tensorial representation of (5); with (9.a) it is immediate to prove that (7.a) and (7.b) imply (2.a) and (2.b), respectively.

From (4,6) is easy to determine the two Lorentz invariants of the electromagnetic field [1,2]:

$$\begin{aligned} I_1 &= F^{ab} F_{ab} = 2(c^2 B^2 - E^2) \quad , \\ I_2 &= *F^{ab} F_{ab} = -4c \vec{B} \bullet \vec{E} \end{aligned} \quad (10)$$

The multiplication of Faraday complex vector [3]:

$$\vec{F} = c\vec{B} + i\vec{E} \quad , \quad (11)$$

with itself leads to (10):

$$\vec{F} \bullet \vec{F} = \frac{1}{2} (I_1 - iI_2) \quad , \quad (12.a)$$

Besides, it gives us another manner to express the Maxwell equations (1):

$$\vec{\nabla} \bullet \vec{F} = 0 \quad , \quad \vec{\nabla} \times \vec{F} = -\frac{1}{c} \frac{\partial \vec{F}}{\partial t} \quad (12.b)$$

Here we shall employ the tensorial formalism to study the eigenvalue problem of (4.b).

## II. FARADAY TENSOR.

The tensorial technique, applied to proper values and eigenvectors of (4.b), implies the equation:

$$F_b^a \xi^b = \lambda \xi^a \quad , \quad (13)$$

which has solution only for  $\lambda$  verifying:

$$\det(\vec{F} - \lambda \vec{I}) = 0 \quad , \quad (14.a)$$

with the corresponding characteristic equation [4-6] :

$$\lambda^4 + \frac{I_1}{2} \lambda^2 - \frac{I_2^2}{16} = 0 \tag{14.b}$$

In this stage it is convenient to introduce the Piña [7]- Sygne [8] classification for  $\tilde{F}$  :

$$\begin{aligned} \text{Type A: } & I_2 \neq 0 \text{ ,} \\ \text{Type B: } & I_2 = 0 \text{ , } I_1 < 0 \text{ ,} \\ \text{Type C: } & I_2 = 0 \text{ , } I_1 = 0 \text{ (Null)} \\ \text{Type D: } & I_2 = 0 \text{ , } I_1 > 0 \text{ ,} \end{aligned} \tag{15}$$

of particular importance in the study of classical charged particles in motion under a constant electromagnetic field [7-10].

The roots of (14.b) follow the scheme:

Type A.- Two real and two imaginaries proper values:

$$\pm \frac{1}{2} \left[ -I_1 + \sqrt{I_1^2 + I_2^2} \right]^{\frac{1}{2}} \text{ , } \pm \frac{i}{2} \left[ I_1 + \sqrt{I_1^2 + I_2^2} \right]^{\frac{1}{2}} \tag{16.a}$$

Type B.- Four real eigenvalues:

$$\lambda_1 = -I_1 \text{ , } \lambda_2 = I_1 \text{ , } \lambda_3 = \lambda_4 = 0 \tag{16.b}$$

Type C.- Collapse of the four proper values:

$$\lambda_j = 0 \text{ , } j = 1, \dots, 4 \tag{16.c}$$

Type D.- Two imaginaries and two real eigenvalues:

$$\lambda_1 = \lambda_2 = 0 \text{ , } \lambda_3 = iI_1 \text{ , } \lambda_4 = -iI_1 \tag{16.d}$$

Type A is our principal aim, with  $\lambda \neq 0$  and the corresponding  $\xi^r$  in (13) is a real eigenvector in Minkowski space, then:

$$\lambda \xi^r \xi_r = F^{ab} \xi_a \xi_b \equiv 0 \text{ [because } F^{ab} \text{ is antisymmetric] ,} \tag{17.a}$$

and as  $\lambda \neq 0$  we conclude that  $\xi^r$  is a null vector:

$$\xi^r \xi_r = (\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 = 0 \quad , \quad (17.b)$$

which means that  $\xi^j$  is on null cone with vertex in the event under study. Therefore, in according with (16.a) we have two independent null eigenvectors:

$$F_b^a \sigma^a = \lambda \sigma^a \quad , \quad F_b^a \eta^b = -\lambda \eta^a \quad , \quad (18.a)$$

$$\sigma^r \sigma_r = \eta^r \eta_r = 0 \quad , \quad \lambda = \frac{1}{2} \left[ -I_1 + \sqrt{I_1^2 + I_2^2} \right]^{\frac{1}{2}} \quad , \quad (18.b)$$

and without loss of generality we select them such that:

$$\sigma^a \eta_a = 1 \quad (18.c)$$

In Linear Algebra [4] it is known that a matrix can be constructed via its proper vectors and eigenvalues, which for the case of Faraday tensor is explicit in the relation [1]:

$$F_{ab} = \lambda (\sigma_a \eta_b - \sigma_b \eta_a) + \tau \eta_{abqr} \sigma^q \eta^r \quad , \quad (19.a)$$

where  $\tau$  has the same sign as  $I_2$ :

$$\tau = \frac{\epsilon}{2} \left[ I_1 + \sqrt{I_1^2 + I_2^2} \right]^{\frac{1}{2}} \quad , \quad \epsilon I_2 > 0 \quad , \quad \epsilon = \pm 1 \quad (19.b)$$

From (18.b,c,19.a) we deduce the expressions (18.a), and the use of (19.a) into (9.a) shows that the Dual tensor is generated by:

$$*F_{ab} = -\tau (\sigma_a \eta_b - \sigma_b \eta_a) + \lambda \eta_{abqr} \sigma^q \eta^r \quad , \quad (20.a)$$

with the same eigenvectors but associated to different proper values:

$$*F_b^a \sigma^b = -\tau \sigma^a \quad , \quad *F_b^a \eta^b = \tau \eta^a \quad (20.b)$$

In [11-14] the relations (18.a,19.a) are shown for the Liénard-Wiechert electromagnetic field [2].

The expressions (19.a,20.a) permit to show algebraic identities for the Faraday matrix and its Dual [7,15,16]:

$$*F^{ab} F_{br} = -\frac{I_2}{4} \delta_r^a \quad , \quad (21.a)$$

$$F^{ab}F_{rb} - {}^*F^{ab}{}^*F_{rb} = \frac{I_1}{2}\delta_r^a \quad , \quad (21.b)$$

$$F_{rb}F^{bq}F_{qa} = \frac{I_1}{2}F_{ar} + \frac{I_2}{4}{}^*F_{ar} \quad , \quad (21.c)$$

which also may be obtained employing (4.a, 6.a). The relations (21) are important to determine the trajectories of charged particles into a uniform electromagnetic field [7-10,15]. From (21.a) it is immediate that:

$$\tilde{F}^{-1} = -\frac{4}{I_2} {}^*\tilde{F} \quad , \quad I_2 \neq 0 \quad , \quad (22.a)$$

that is, the Dual matrix give us the inverse of  $\tilde{F}$  when  $I_2 \neq 0$  , which exhibits a possible relationship between this invariant and the determinant of  $\tilde{F}$  , in fact, the characteristic equation (14.b) implies [5,6]:

$$\det \tilde{F} = -\frac{I_2^2}{16} = -\left(c\vec{B} \cdot \vec{E}\right)^2 \quad , \quad (22.b)$$

then exists  $\tilde{F}^{-1}$  in the points where  $\vec{B}$  is not orthogonal to  $\vec{E}$  . From (21.a, c) result the identity:

$$\tilde{F}^4 + \frac{I_1}{2}\tilde{F}^2 - \frac{I_2^2}{16}\tilde{I} = \tilde{0} \quad , \quad (23)$$

equivalent to Cayley–Hamilton theorem [4], thus  $\tilde{F}$  satisfies its characteristic equation (14.b).

Type C showed in (15) corresponds to a null electromagnetic field:

$$I_1 = I_2 = 0 \quad , \quad (24)$$

and thus (16.c) implies only the zero as eigenvalue:

$$F_b^a K^b = 0 \quad , \quad (25.a)$$

then it is possible to prove [1,17] that  $K^r$  is a null eigenvector of Dual tensor:

$${}^*F_b^a K^b = 0 \quad , \quad (25.b)$$

or equivalently:

$$F_{ab}K_r + F_{br}K_a + F_{ra}K_b = 0 \quad (25.c)$$

The following expression [1,17] permits the construction of  $\tilde{F}$  :

$$\begin{aligned} F_{ab} &= K_a Z_b - K_b Z_a \quad , \quad {}^*F_{ab} = \eta_{abpq} K^p Z^q \quad , \\ K_r Z^r &= 0 \quad , \quad Z^r Z_r = -1 \quad , \end{aligned} \tag{25.d}$$

where the unitary space-like vector  $Z^r$  is non-unique, in fact, it is clear that in (25.d) we may add to  $Z^r$  the vector  $\beta K^r$  , being  $\beta$  an arbitrary parameter, without to alter such relations.

From (21,24) we obtain the following identities valid for any  $\tilde{F}$  with Type C:

$${}^*\tilde{F}\tilde{F} = \tilde{0} \quad , \quad {}^*F^2 = F^2 \quad , \quad \tilde{F}^3 = \tilde{0} \quad , \tag{26}$$

which also are obtainable from (25.d).

The real null vector  $K^b$  is a Null Principal Direction of  $\tilde{F}$  , with Type C, because it verifies the properties (25.a, b):

$$W_{ab} K^b = 0 \quad , \tag{27.a}$$

with the complex antisymmetric tensor:

$$W_{ab} = F_{ab} + i {}^*F_{ab} \quad , \quad i = \sqrt{-1} \tag{27.b}$$

The vectors  $\sigma^a$  and  $\eta^a$  , for the Type A, are Null Principal Directions of  $\tilde{F}$  because they does not satisfy (27.a) but they verify [17]:

$(\xi_r W_{ab} - \xi_a W_{rb}) \xi^b = 0 \quad , \quad \xi^r = \sigma^r \text{ and } \eta^r \quad ,$	(27.c)
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which it is immediate from (18,19,20).

We stress that in (25.a,b) the null vector  $K^b$  is unique (except for a scale factor) because there is not two independent null eigenvectors for the proper value zero. Besides, after the selection of  $Z^r$  in (25.d), then is unique the unitary space-like proper vector  $Y^r$  :

$$F_b^a Y^b = 0 \quad , \quad Y^r Z_r = Y^r K_r = 0 \quad , \quad Y^r Y_r = -1 \quad , \tag{28.a}$$

then the Dual tensor adopts the form:

$${}^*F_{ab} = K_a Y_b - K_b Y_a \quad , \tag{28.b}$$

therefore  $Z^r$  is a space-like eigenvector of  ${}^* \tilde{F}$ , but not of  $\tilde{F}$ :

$${}^* F_b^a Z^b = 0 \quad F_b^a Z^b = -K^a \tag{28.c}$$

This work is principally dedicated to Faraday tensor, however, it is interesting to realize an application to energy-momentum of Maxwell defined by [1,2,18]:

$$T^{ab} = -F^{ar} F_r^b + \frac{I_1}{4} g^{ab} \quad , \tag{29.a}$$

which is symmetric due to relativist equivalence between mass and energy, and it has zero trace because the photon has null rest mass:

$$T^{ab} = T^{ba} \quad , \quad T_a^a = 0 \tag{29.b}$$

Then for the Type A we multiply (29.a) by  $\sigma_b$  and  $\eta_b$ , and we use (18.a), thus we deduce that these proper vectors of  $\tilde{F}$  and  ${}^* \tilde{F}$  also they are eigenvectors of Maxwell tensor:

$$T_b^a \xi^b = \frac{1}{2} (\lambda^2 + \tau^2) \xi^a \quad , \quad \xi^r = \sigma^r \text{ and } \eta^r \quad , \tag{30.a}$$

and with (18.b, 19.b) we obtain the relations:

$$\tau^2 - \lambda^2 = \frac{I_1}{2} \quad , \quad \lambda\tau = \frac{I_2}{4} \quad , \quad \lambda^2 + \tau^2 = \frac{1}{2} \sqrt{I_1^2 + I_2^2} \tag{30.b}$$

From (18.c, 19.a, 30.b) into (29.a) it is possible to write  $T^{ab}$  in terms of its null eigenvectors and Minkowski metric  $g^{ab} = \text{Diag}(1,-1,-1,-1)$ :

$$T^{ab} = (\lambda^2 + \tau^2) \left( \sigma^a \eta^b + \sigma^b \eta^a - \frac{1}{2} g^{ab} \right) \tag{30.c}$$

For the Type C the use of (24,25.d) in (29.a) leads to radiative Maxwell tensor:

$$T^{ab} = K^a K^b \quad , \tag{31.a}$$

and with (25.d,28.a) we can show its proper vectors:

$$T_b^a K^b = T_b^a Z^b = T_b^a Y^b = 0 \tag{31.b}$$

From (30.c,31.a) are evident the properties (29.b), and also the important Rainich conditions [19-22]:

$$T^{ab}T_{rb} = \frac{1}{4}(T^{mn}T_{mn})\delta_r^a \quad , \quad (32.a)$$

where:

$$T^{mn}T_{mn} = (\lambda^2 + \tau^2)^2 = \frac{1}{4}(I_1^2 + I_2^2) \quad (32.b)$$

It is interesting to see that a phase change in the Faraday complex vector (11):

$$c\tilde{\vec{B}}+i\tilde{\vec{E}}=e^{i\xi}(c\vec{B}+i\vec{E}), \quad (33.a)$$

which is equivalent to Duality Rotations [16,19, 20, 23, 24] :

$$\begin{pmatrix} \tilde{\vec{E}} \\ c\tilde{\vec{B}} \end{pmatrix} = \begin{pmatrix} \cos \xi & \text{sen} \xi \\ -\text{sen} \xi & \cos \xi \end{pmatrix} \begin{pmatrix} \vec{E} \\ c\vec{B} \end{pmatrix}, \quad (33.b)$$

induces the same rotation in the matrices  $\tilde{F}$  and  $*\tilde{F}$  :

$$\begin{pmatrix} \tilde{F}^{ab} \\ *F^{ab} \end{pmatrix} = \begin{pmatrix} \cos \xi & \text{sen} \xi \\ -\text{sen} \xi & \cos \xi \end{pmatrix} \begin{pmatrix} F^{ab} \\ *F^{ab} \end{pmatrix} \quad , \quad (33.c)$$

but it generates a rotation of double angle in the invariants (10):

$$\begin{pmatrix} \tilde{I}_1 \\ \tilde{I}_2 \end{pmatrix} = \begin{pmatrix} \cos(2\xi) & \text{sen}(2\xi) \\ -\text{sen}(2\xi) & \cos(2\xi) \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \quad (33.d)$$

If now we employ (33.c, d) in (29.a) results a remarkable fact, the Maxwell tensor is invariant under Duality rotations:

$$\tilde{T}^{ab} = T^{ab} \quad , \quad (34.a)$$

besides, this tensor has connection with the electromagnetic energy-momentum, therefore it is natural to think that the invariance (34.a) may have relationship with some conservation law. The equations (1) can be deduced from a variational principle [25], then the Noether theorem [25-27] for the transformations (33.a) implies the continuity equation [18]:

$$\frac{\partial}{\partial t} \left( \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right) + \vec{\nabla} \cdot \left( \frac{1}{\mu_0} \vec{E} \times \vec{B} \right) = 0 \quad , \quad (34.b)$$

for the Poynting vector and the electromagnetic energy density.

### III. CONCLUSION

The present work shows that the study of algebraic structure of Faraday tensor gives us a better understanding of Maxwell field.

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