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On the Fueter-Lanczos Conditions

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ABSTRACT.- We show that the Maxwell equations in vacuum and the Dirac equation without the mass term are special cases of the Fueter-Lanczos conditions.
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KEYWORDS: Quaternions; Maxwell equations; Fueter-Lanczos relations; Dirac equation

I. INTRODUCTION

In [1] p.453 we find that the Cauchy-Riemann conditions [2] can be written in the form:

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(u+iv)=\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+i\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0\,,\quad i=\sqrt{-1} \tag{1}$$

We may remember that in complex variable an analytic function f(z) = u + i v must verify (1), which implies the harmonic character of real and imaginary parts of f(z), that is, u & v satisfy the Euler (1752)-Laplace (1787) equation in two dimensions, and this fact has great physical importance because it permits to apply the Conformal Mapping [2,3] to many problems from electrostatics, hydrodynamics, thermostatic, etc., which obey the Euler-Laplace equation. Besides, in [4] were employed the relations (1) to show that u & v are Debye auto-potentials, in other words, the famous electromagnetic Debye potentials [5-9] may be motivated through Cauchy-Riemann conditions. The multiplication of z by exp (ib) produces its rotation by an angle b in the Argand plane, then it is immediate to ask if there is a system of numbers generating rotations in three (our Euclidean space) and four (Minkowski space-time) dimensions, in fact, these system corresponds to Quaternions discovered by Hamilton the 16th Oct. 1843 [10-14]. In [15-21] we may find the quaternionic process to generate 3-rotations and Lorentz transformations.

It was mentioned that the complex relation (1) has impact in several physical applications, thus it is natural to search its corresponding quaternionic generalization, and in this quest it is relevant the form (1) given by Lanczos [1] to Cauchy-Riemann conditions, because thus it is direct its extension via quaternions:

$$\nabla\, G = 0 \tag{2}$$

where:

$$\nabla = I\,\frac{\partial}{\partial x_1}+J\,\frac{\partial}{\partial x_2}+K\,\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4}\qquad , \tag{3.a}$$

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$$G = I\,u_1 + J\,u_2 + K\,u_3 + u_4 \tag{3.b}$$

Using (3.a,b) in (2) we deduce the Fueter conditions [22]:

$$\begin{aligned}
&\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}-\frac{\partial u_4}{\partial x_4}=0 \quad ,\\
&\frac{\partial u_4}{\partial x_1}+\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}+\frac{\partial u_1}{\partial x_4}=0 \quad ,\\
-&\frac{\partial u_3}{\partial x_1}+\frac{\partial u_4}{\partial x_2}+\frac{\partial u_1}{\partial x_3}+\frac{\partial u_2}{\partial x_4}=0 \quad ,\\
&\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}+\frac{\partial u_4}{\partial x_3}+\frac{\partial u_3}{\partial x_4}=0 \quad ,
\end{aligned} \tag{4}$$

first obtained by Lanczos [1] eqs. (21.a). We note that in complex variable the relations (1) have connection with the existence of the derivative of function u + i v, however, in quaternionic variable the Fueter-Lanczos expressions (4) must be interpreted as Regularity Conditions [23] of (3.b). Similar to (1), the restrictions (4) can be written [19,24] in the Debye form, which motivates a connection between (4) and Maxwell equations, to see Sec. 2. Imaeda [23] shows that Fueter-Lanczos conditions permit a new formulation of classical electrodynamics. We commented that the quest of techniques to produce rotations in three and four dimensions leads to quaternions, and therefore to Fueter-Lanczos relations, and it is a great surprise to find that (4) have relationship with other types of rotations (or angular momentum): Faraday vector appears in Sec. 2 because the Maxwell equations in vacuum are invariant under Duality Rotations [9,15,19,25-29], and these vector gives us guidance to select the function (3.b) such that (2) implies the Maxwell equations in free space; besides, in Sec. 3 we have other type of rotation, the spin of electron, because another election of (3.b) makes that (2) [or equivalently (4)] coincides with Dirac equation for spin 1/2 [30] without the mass term. Thus it is remarkable the great amount of information, about rotations, into quaternionic formalism. Lanczos [1,31-33] employed quaternions to realize a profound analysis of Dirac equation, and thus he also obtained the Proca-Kemmer equation for spin 1 [34,35] and the idea of Isospin. The Ref. [36] has an interesting historical study of Dirac equation.

In resumé, in Secs. 2 and 3 we consider particular cases of (3.b) to exhibit that (4) imply the Maxwell equations in vacuum and Dirac equation without the mass term, respectively. In the deduction of (4) were useful the known rules of multiplication for the quaternionic units [20,21]:

$$I^2 = J^2 = K^2 = -1, \qquad I\,J\,K = -1 \tag{5}$$

II. MAXWELL EQUATIONS WITHOUT SOURCES

In free space the electromagnetic field follows the differential equations [c is the light velocity in vacuum]:

$$\vec{\nabla} \bullet \vec{B} = 0 \quad , \qquad \vec{\nabla} \bullet \vec{E} = 0 \quad ,$$
$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \quad , \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad , \tag{6}$$

for the electric and magnetic 3-vectors. If now we select:

$$x_1 = x \ , \ x_2 = y \ , \ x_3 = z \ , \ x_4 = -ict \quad , \tag{7}$$

and

$$u_1 = cB_x + iE_x \quad , \ u_2 = cB_y + iE_y \quad ,$$
$$u_3 = cB_z + iE_z \ , \ u_4 = 0 \quad , \tag{8}$$

then the relations (4) reproduce (6). That is, the Maxwell equations in vacuum adopt the form (2) [25,37]:

$$\nabla F = 0 \quad , \tag{9}$$

where

$$\nabla = I\frac{\partial}{\partial x} + J\frac{\partial}{\partial y} + K\frac{\partial}{\partial z} + \frac{i}{c}\frac{\partial}{\partial t} \ , \tag{10.a}$$

$$F = IF_x + JF_y + KF_z \ , \vec{F} = c\vec{B} + i\vec{E} \tag{10.b}$$

first obtained by Lanczos [1,38] and Silberstein [39]. In (10.b) participates the Faraday complex vector [15,19,25] because the system (6) is invariant under Duality Rotations. It is interesting to mention that the quaternionic expression (9) permits to deduce, in natural manner, the corresponding spinorial form of Maxwell equations [25], thanks to the relationship between quaternionic units and Pauli matrices [41].

III. DIRAC EQUATION

Here we consider the Fueter-Lanczos conditions for (7) with:

$$u_1 = \overline{\psi}_2 + \psi_2 + \overline{\psi}_4 - \psi_4 \ , u_2 = i\left(\overline{\psi}_2 - \psi_2 + \overline{\psi}_4 + \psi_4\right),$$
$$u_3 = \overline{\psi}_1 + \psi_1 + \overline{\psi}_3 - \psi_3 \ , u_4 = i\left(\overline{\psi}_1 - \psi_1 + \overline{\psi}_3 + \psi_3\right), \tag{11}$$

then, from (7,11) in (4), we find that the functions ψ_j must satisfy the system of differential equations:

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_4+\frac{\partial \psi_3}{\partial z}+\frac{1}{c}\frac{\partial \psi_1}{\partial t}=0 \quad ,$$

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_3-\frac{\partial \psi_4}{\partial z}+\frac{1}{c}\frac{\partial \psi_2}{\partial t}=0 \quad ,$$

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_2+\frac{\partial \psi_1}{\partial z}+\frac{1}{c}\frac{\partial \psi_3}{\partial t}=0 \quad , \qquad (12)$$

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_1-\frac{\partial \psi_2}{\partial z}+\frac{1}{c}\frac{\partial \psi_4}{\partial t}=0 \quad ,$$

and it is equivalent to Dirac equation for spin $\frac{1}{2}$ without the mass term [30,36,42], which takes the form (2) :

$$\nabla\, D = 0 \quad , \qquad (13)$$

with ∇ and D given by (10.a) and (3.b, 11), respectively. The expressions (9) and (13) exhibit a remarkable connection, in their structure, between Maxwell and Dirac equations, which was showed by Lanczos [1,38], Darwin [43] and Rainich [44]. Other notable aspect of quaternionic formalism is that it permits to construct [16,45], in simple and direct manner, the 16 Dirac matrices of relativistic quantum mechanics [42].

The relations (4) participate in general relativity: Lanczos [46] showed that, in any curved space-time, the conformal Weyl tensor is generated by a potential of 3th order, named Spintensor because in weak gravitational fields he obtained (13), and his expectation was the usefulness of these potential in the unification of gravity and quantum mechanics.

IV. CONCLUSIONS

This brief work exhibits the importance in Theoretical Physics of Fueter-Lanczos conditions.

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