JVR

Journal of Vectorial Relativity
JVR **4** (2009) **3** 129-132

Letter

On the Daubechies Polynomials

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ABSTRACT. We show that with small modifications in diverse expressions for the shifted Legendre polynomials, we can obtain the corresponding relations for Daubechies polynomials.

KEYWORDS: Shifted Legendre polynomials; Daubechies polynomials

I. INTRODUCTION

The wavelets are very important in science, engineering and technology [1-3], in particular, the construction of Daubechies wavelets [4] depends strongly from zeros of Daubechies polynomials d_l [5], thus it is interesting to study the properties of these polynomials because its behavior gives us useful information on the corresponding wavelets. Under this approach, here we show that the analysis of d_l may be guided through the modified Legendre polynomials $P_l{}^*$ [6-8], therefore a better understanding of Daubechies polynomials can be obtained via the Legendre functions.

II. SHIFTED LEGENDRE POLYNOMIALS

In fact, shifted Legendre polynomials [6-8], for $x \in [0,1]$:

$$P_0^* = 1, \qquad P_1^* = 1 - 2x, \quad P_2^* = 1 - 6x + 6x^2,$$
$$p_3^* = 1 - 12x + 30x^2 - 20x^3, \tag{1}$$
$$P_4^* = 1 - 20x + 90x^2 - 140x^3 + 70x^4$$
$$P_5^* = 1 - 30x + 210x^2 - 560x^3 + 630x^4 - 252x^5, \text{ etc.}$$

are solutions of the differential equation:

$$x(1-x)y'' - (2x-1)y' + l(l+1)y = 0 \tag{2}$$

and they can be generated with the expression:

$$P_l^*(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} \binom{l+k}{k} x^k, \tag{3}$$

or in terms of Gauss hypergeometric function [7,8]:

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$$P_l^*(x) = {}_2F_1(-l, l+1; 1; x) \tag{4}$$

III. SIMILARITIES BETWEEN SHIFTED LEGENDRE POLYNOMIALS AND DAUBECHIES POLYNOMIALS

By making a sign change into 2nd coefficient of (2) to consider the slightly different equation:

$$x(1-x)y'' - (2x+l)y' + l(l+1)y = 0 \tag{5}$$

it is nice to discover that the Daubechies polynomials [5], for $|x| \le 1$:

$$\begin{gathered} d_0 = 1, \quad d_1 = 1 + 2x, \; d_2 = 1 + 3x + 6x^2, \\ d_3 = 1 + 4x + 10x^2 + 20x^3, \quad d_4 = 1 + 5x + 15x^2 + 35x^3 + 70x^4, \\ d_5 = 1 + 6x + 21x^2 + 56x^3 + 126x^4 + 252x^5, \text{ etc.,} \end{gathered} \tag{6}$$

are solutions of (5). **Fig. 1** shows the polynomials (6), where only we can see their real roots; in general, their zeros are complex, for example, roots of $\boldsymbol{d_6}$ are (*0.1411+0.3421 i*), (*-0.1246+0.2832 i*), (*-0.2665+0.1073 i*) and their conjugates. Besides, there we note that $\boldsymbol{d_l}(0)=1$ and $\boldsymbol{d_l}(x) > 0$ if $x>0$, thus the real zeros are negative.

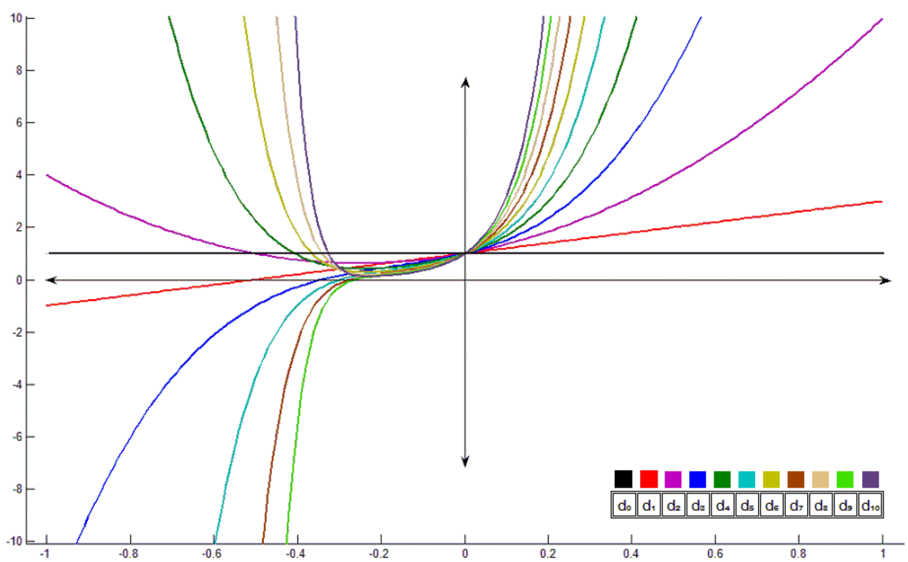

Figure 1.- Daubechies polynomials.

The d_l are very important in the construction of the compactly supported Daubechies wavelets. There is a close relationship between the zeros of d_l and the 2*l* filter coefficients *h*(*l*) of the Daubechies wavelets D_{2l} [5]. Therefore, it is fundamental to search efficient algorithms to find the roots of Daubechies polynomials, especially for large *l.* Here we exhibit certain connections between (1) and (6), and then we hope that the stored experience with the roots of Legendre functions may be useful in the analysis of the zeros of (6).

It is easy to find the corresponding modification of (4):

$$d_l(x) = \lim_{\lambda \to 0} {}_2F_1(-l, l+1; -l+\lambda; x) \tag{7}$$

and thus (3) adopts the known form [5]:

$$d_l(x) = \sum_{k=0}^{l} \binom{l+k}{k} x^k \tag{8}$$

Therefore, in the equation:

$$(1-x)y'' - (2x+a)y' + l(l+1)y = 0 \tag{9}$$

we may indicate two cases of interest:

$$y(x) = \begin{cases} P_l^x, & a = -1, \quad 0 \le x \le 1 \\ d_l(x), & a = l, \quad |x| \le 1 \end{cases} \tag{10}$$

with the following Rodrigues formulae:

$$P_l^x(x) = \frac{1}{l!}\frac{d^l}{dx^l}[x(1-x)]^l \tag{11.a}$$

$$d_l(x) = \frac{1}{l!}\left(\frac{x}{x-1}\right)^{l+1}\frac{d^l}{dx^l}\left[\frac{(x-1)^{2l+1}}{x}\right] \tag{11.b}$$

which generate to (1) and (6). The expression:

$$y(x) = \frac{\left[b + l + (1-b)(-1)^l\right]}{(l+1)!}\left(\frac{x}{x-1}\right)^{b+l}\frac{d^l}{dx^l}\left[\frac{(x-1)^{b+2l}}{x^b}\right], \tag{12}$$

for $b = -l$ and $b = 1$ reproduces (11.a) and (11.b), respectively.

The relation (8) may be written as:

$$d_l(x) = \sum_{k=0}^{l} d_{lk} x^k, \qquad d_{lk} = \binom{l+k}{k}, \qquad k = 0,1,\ldots,l \tag{13}$$

then it is immediate to obtain that:

$$d_{l0} = 1, \qquad d_{lj} = \sum_{k=0}^{j} d_{l-1}, \qquad j = 1,\ldots,n-1 \tag{14}$$

$$d_{ll} = 2d_{l,\ \ l-1}$$

which means that the coefficients of $d_{l-1}(x)$ permit to construct the next $d_l(x)$. Finally, we note the following property of Daubechies polynomials:

$$(1-x)^{l+1}\ d_l(x) + x^{l+1} d_l(1-x) = 1 \tag{15}$$

IV. CONCLUSION

The intention of this Note was to show the parallelism between shifted Legendre and Daubechies polynomials.

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