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Gravitational and Inertial Mass in Vectorial Relativity

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ABSTRACT: This work removes the eternal controversy between gravitational and inertial mass by defining mass as unique, and by establishing a clear difference between the definitions of **Gravitational Field** $\mathcal{G}$ produced by a massive body M, appearing in the Universal Law of Gravitation, $\vec{\mathcal{F}} = m.\vec{\mathcal{G}}$, and the **acceleration** $a = \frac{dv}{dt}$ of a moving mass m appearing in the second Newton's Law, $\vec{\mathcal{F}} = \frac{d(m.\vec{v})}{dt}$. A rundown of foundations of Vectorial Relativity is presented for obtaining the rectilinear $\mathcal{G}_R$ and curvilinear $\mathcal{G}_C$ expressions of the Gravitational Field as they were originally obtained in previous work. Additionally, other expressions were developed in order to reveal kinematical properties of Gravitational Field that agree known experimental results.

KEYWORDS: Universal Gravitation Law, Inertial and gravitational mass, Equivalence Principle, Vectorial Relativity.

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I. INTRODUCTION

Isaac Newton realized the mutual attraction between all masses and particles of matter. Trying to explain the general concept of gravity in 1686 established that the motion of stars and planets, and moon, and sun, and each body in the universe, as well as that of the falling apple, could be explained by his Law of Universal Gravitation, which stated something like this: "*any two objects attract each other with a force that acts in the line joining them, such that its intensity varies directly as the product of their masses and inversely as the square of the distance between their centers of mass, times a constant of proportionality G*", the currently well-known constant of Gravitation.

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During eighteenth and nineteenth centuries gravitational astronomy, based on Newton's Universal Gravitation Law and Kepler's Laws (a straightforward consequence of Newton's Laws), explained wonderfully every question about motion of celestial bodies. Nevertheless, this peace was only apparent. Since the very beginning of Newton's proposal, physicists were immersed until today in a very deep discussion along these more than 300 years, about the determination and verification of the gravitational constant, the not accounting in Newtonian gravitation of advances of perihelion in Mercury's orbit, the validity of Ether concept, the different meanings of inertial and gravitational mass, the influence of frames of reference in Physics, etc. In this work we access again some of these aspects under the optics of the Vectorial Relativity Theory.

This work provides an introduction to the modeling of gravitation by taking mass varying with velocity, or in a more general way, varying with time. In this sense we have made an effort to keep presentation pedagogical.

Throughout the article we will assume that Relativity is the proper description of nature in the sense of mass variation, though we don't say anything about what kind of Relativity in order to allow the possibility of application of any theory. As we hope that the article will be used by students and researchers who are not necessarily experts in relativity, we have developed the discussion with the simplest mathematical tools required to build models of relativistic objects. Given the potential for future applications of this formalism, we have opted to base much of our description on previous work on the same topic [1].

II. SUMMARY OF VECTORIAL RELATIVITY

Let's present a resume of the sequence of concepts that build the infrastructure of Vectorial Relativity.

This work began in 1995 with some innocuous and ingenuous questions about the foundations of Relativity: Why the Lorentz Transformations (LT) are not presented in a general manner, in the sense that when trying to look at the configuration of the two observers located on distinct inertial systems with relative motion between them, in order to study the posing of the problem in the literature referred to this subject always the moving observer is confined to move along one of the axis? Why it is not possible to obtain a general treatment in any text that deals with this subject?. These reflections motivated our decision to obtain the Lorentz Transformations between the two observers, by posing the problem in a more general configuration in the sense of allowing the second observer to move in the space along a generic inclined line m**,** with respect to any of the axes. After some calculations other expressions different from the known LT, were obtained (!). What did this mean?. Was this configuration adequate to pose the problem of looking for the transformations between an observer located on a fixed system and another observer located on a moving system?

After doing checks and recalculations our conclusion was that our calculations were correct and our configuration was adequate to obtain the general transformations between two inertial observers, that move relative each other at speed v.

In Fig. 1 is depicted the configuration that allows constructing the necessary relationships for obtaining the Lorentz Transformations which will serve us to develop Relativity.

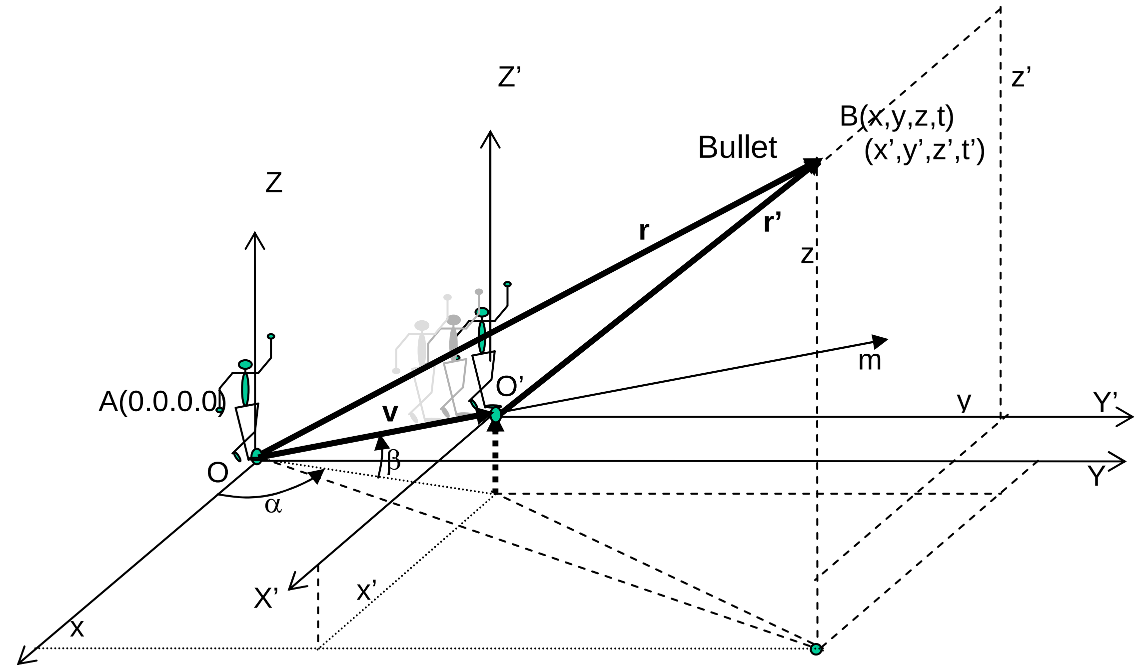


Fig. 1 **Three-dimensional motion**

For example, it was found that the equation $x' = k.(x - v.t)$ holds good, but equations, $y' = y$, $z' = z$, were not really transformations that arise from the conditions on which the LT were based but incorrect assumptions, according to obtained results. If this was a surprise, as it was, the shock during these first steps was to realize that, without doing any kind of assumptions, **time displayed itself as a common and ordinary vector with components on the three spatial coordinates**. In this way, I was able to find the Generalized Lorentz Transformations, which I named Vectorial Lorentz Transformations (VLT) to emphasize the surprising vector character of time revealed as result of equations, for instance:

In one-dimensional space, the resulting interpretation of equations inside squares is:

$$x = c.t \qquad \boxed{x' = k.(x - v.t)} \qquad \boxed{t' = k.(t - \frac{v}{c^2}x)} \qquad \Rightarrow \qquad \begin{aligned} \mathbf{t}' &= k.(\mathbf{t} - \frac{v}{c^2}.\mathbf{r}) \\ \mathbf{r}' &= k.(\mathbf{r} - v.\mathbf{t}) \end{aligned}$$

For a two-dimensional space, derivation in polar coordinates (less direct) following equations hold:

$$\begin{aligned} x^2 + y^2 &= c^2.t^2 \\ x'^2 + y'^2 &= c^2.t'^2 \end{aligned} \qquad \boxed{\begin{aligned} x' &= k.(x - v.t.\cos\alpha) \\ y' &= k.(y - v.t.\sin\alpha) \end{aligned}}$$

Where, α, is the angle between trajectory of O' and X axis; By defining time components as $t_x = t.\cos\alpha$, $t_y = t.\sin\alpha$, we obtain:

$$\boxed{\begin{aligned} x' &= k.(\,x - v.t_x\,) \\ y' &= k.(\,t - v.t_y\,) \end{aligned}}$$

Substituting and grouping:

$$c^2.t'^2 = x'^2 + y'^2 = k^2.[(x - v.t_x)^2 + (y - v.t_y)^2]$$

$$c^2.t'^2 = k^2.[x^2 + y^2 + v^2.t_x^2 + v^2.t_y^2 - 2.v.x.t_x - 2.v.y.t_y]$$

$$c^2.t'^2 = k^2.[c^2.t^2 + v^2.t^2 - 2.v.(x.t_x + y.t_y)]$$

$$c^2.t'^2 = k^2.[c^2.(t_x^2 + t_y^2) + v^2.\frac{x^2 + y^2}{c^2} - 2.v.(x.t_x + y.t_y)]$$

$$c^2.t'^2 = k^2.[(c.t_x - \frac{v}{c}.x)^2 + (c.t_y - \frac{v}{c}.y)^2]$$

Dividing by c^2, a vectorial structure of time is obtained and follows, for expression inside squares:

$$\boxed{t'^2 = k^2.[(t_x - \frac{v}{c^2}.x)^2 + (t_y - \frac{v}{c^2}.y)^2]} \quad \Rightarrow \quad \begin{aligned} \mathbf{t}' &= k.(\mathbf{t} - \frac{v}{c^2}.\mathbf{r}) \\ \mathbf{r}' &= k.(\mathbf{r} - v.\mathbf{t}) \end{aligned}$$

For the three-dimensional case, from Fig. 1 the following relationships hold:

$$\begin{aligned} x^2 + y^2 + z^2 &= c^2.t^2 \\ x'^2 + y'^2 + z'^2 &= c^2.t'^2 \end{aligned} \qquad \boxed{\begin{aligned} x' &= k.(x - v.t.\cos\alpha.\cos\beta) \\ y' &= k.(y - v.t.\sin\alpha.\cos\beta) \\ z' &= k.(z - v.t.\sin\beta) \end{aligned}} \qquad \begin{aligned} t_x &= t.\cos\alpha.\cos\beta \\ t_y &= t.\sin\alpha.\cos\beta \\ t_z &= t\sin\beta \end{aligned}$$

Going by a similar procedure to that previously used, it is obtained again the familiar vector structure of time for three (or, extrapolating, for any number of) dimensions:

$$\boxed{t'^2 = k^2.[(t_x - \frac{v}{c^2}.x)^2 + (t_y - \frac{v}{c^2}.y)^2 + (t_z - \frac{v}{c^2}.z)^2]} \quad \Rightarrow \quad \begin{aligned} \mathbf{t}' &= k.(\mathbf{t} - \frac{v}{c^2}.\mathbf{r}) \\ \mathbf{r}' &= k.(\mathbf{r} - v.\mathbf{t}) \end{aligned}$$

All these results lead consistently to consider **time as a vector** when it is referred to observers located in systems with distinct inertial movements.

The basic set of corrected Lorentz transformations or Vectorial Lorentz Transformations (VLT) for kinematics (branch of mechanics which describes the motion of objects without the consideration of the masses or forces) is:

$$x'=\frac{x-v.t.\cos\alpha.\cos\beta}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \qquad x'=\frac{x-v.t_x}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$y'=\frac{y-v.t.\sin\alpha.\cos\beta}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \begin{aligned} t_x &= t.\cos\beta.\cos\alpha \\ t_y &= t.\cos\beta.\sin\alpha \\ t_z &= t\sin\beta \end{aligned} \qquad y'=\frac{y-v.t_y}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1}$$

$$z'=\frac{z-v.t.\sin\beta}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \qquad z'=\frac{z-v.t_z}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t'=\left|\frac{\mathbf{t}-\frac{v}{c^2}.\mathbf{r}}{\sqrt{1-\frac{v^2}{c^2}}}\right|=\sqrt{\frac{(t_x-\frac{v}{c^2}.x)^2+(t_y-\frac{v}{c^2}.y)^2+(t_z-\frac{v}{c^2}.z)^2}{1-\frac{v^2}{c^2}}} \tag{2}$$

In this way, the new transformations correct the well-known LT. As it can be realized the corrected Lorentz Transformations are symmetric, all dimensions are contracted or expanded by the same factor. By reducing to the situation where observers will do their measurements taking the same point of reference (origin of the fixed observer), the expressions of the Local Lorentz Transformations (LLT) were obtained [1]. The same reference of measurements simplify transformations to be related only through a factor $\left(1-\frac{v^2}{c^2}\right)^N$, for values of the exponent that depend on the physical magnitude. The real physical meaning of LLT can be imagined as if you were fixed at a point and you saw a stick of length L, or a disk of area S or a ball of volume V displacing in the air at a speed v. Additionally, let you previously to know that, at rest, the length of the stick was L_0, the area of the disk was S_0 and the volume of the ball was V_0, which are the dimensions measured by an observer mounted on the stick, the disk or the ball in the air, displacing at a similar speed v. When you are looking at the space and unexpectedly you recognize the known stick, or disk or ball flying in the air you will see them contracted according to the following factors:

$$L_0=\frac{L}{\sqrt{1-\frac{v^2}{c^2}}};\Rightarrow\ L=L_0.\sqrt{1-\frac{v^2}{c^2}}; \qquad S_0=\frac{S}{1-\frac{v^2}{c^2}}\Rightarrow S=S_0.\left(1-\frac{v^2}{c^2}\right)$$

$$V_0=\frac{V}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\Rightarrow V=V_0.\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}} \tag{3}$$

Also, if you see a clock flying in the air, accompanying the stick, the disk or the ball at the same speed v you would realize that your time t goes slower than that of the flying clock (t_0).

$$t_0 = t.\sqrt{1-\frac{v^2}{c^2}} \quad \Rightarrow \quad t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} \tag{4}$$

In Vectorial Relativity everything is symmetric and normal, in the sense that lengths, areas and volumes contracts uniformly, time delays without surprises or if instead the physical magnitude expands, it does it homogeneously. This comment is referred to the complexities observed in the Special Theory of Relativity, with different transversal or longitudinal factors for each physical quantity.

By combining the previously obtained transformations it can be easily achieved all the other definitions in Kinematics, for example, vector velocity transforms as $\mathbf{v}' = \dfrac{\mathbf{v}}{1-\frac{v^2}{c^2}}$ and vector acceleration does as $\mathbf{a}' = \dfrac{\mathbf{a}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}$. Angle becomes invariant because its definition is arc divided by radius (division of two lengths cancels their transformations) $\boldsymbol{\alpha}' = \boldsymbol{\alpha}$. See [1].

After analyzing Kinematics, the obtained most important Local Lorentz Transformation (LLT) was that of relativistic mass and with it the development of the transformations for Dynamics, the branch of mechanics that is concerned with the effects of forces on the motion of objects. The importance of this is very relevant because it corrects an error carried out along one hundred years, since Einstein in 1905, coined the erroneous definition of relativistic mass as: $m = \dfrac{\overset{0}{m}}{\sqrt{1-\frac{v^2}{c^2}}}$. The correct and unique definition of relativistic mass, demonstrated in [1] was:

$$m = \frac{\overset{0}{m}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \tag{5}$$

This definition modifies all Relativistic Dynamics known until now, because all quantity definitions in Dynamics depend on the definition of mass. In this way, new definitions of Energy turn out to be [2]:

$$E = K + \overset{0}{m}.c^2 = m.\left(2.v^2 - c^2\right) + \overset{0}{m}.c^2 + \overset{0}{m}.c^2 = 2.\overset{0}{m}.c^2 - m.\left(c^2 - 2.v^2\right) \tag{6}$$

Linear momentum,

$$p = m.v = \frac{\overset{0}{m}.v}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \tag{7}$$

The next achievement obtained in Vectorial Relativity through the LLT was the invariance of charge, $q' = q \;\Rightarrow\; q = q_0$, and with this all the LLT's of all the electromagnetic quantities [1].

III. GRAVITATION FOR CURVILINEAR MOTION IN VECTORIAL RELATIVITY.

Of course, all these new vectorial relativistic definitions lead directly to check its compatibility with Newton's Universal Law of Gravitation. In doing this and as it have been stated in the abstract of this work, this approach correct the weak points of Newton's Universal Law of Gravitation. As it was concluded in [3], it was required a modification of Newton's definition of Gravitation Force in order to adapt its concept to a mass-variation relativistic model. It is necessary to say that the expression of the Universal Law of Gravitation as we know it, $F = \frac{G.M.m}{r^2}$, for M fixed and m moving at a variable speed around M, which implies that m is also variable, does not meet the Angular Momentum Conservation Law, which is one of the weak points of this definition.

So, this development started with the consideration of a generic variable mass, without establishing any cause of its variation. Later, Newton's principle of attraction of masses was indirectly used for arriving at an expression of a Universal Gravitation Law in agreement with the relativistic point of view of Vectorial Relativity. In this way, it was obtained an expression of the relativistic Gravitational Field that meets the Law of conservation of the Angular Momentum for variable masses in an Isolated System.

The obtained expression of the Gravitational Field generated by a mass M, located in a fixed frame, influencing another generic mass m moving curvilinear around M at variable speed v became [3]:

$$\mathcal{G}_C = \frac{\frac{2.G.M}{r^2}.\frac{v}{V_0} - v.\frac{dv}{dr}.\left(\frac{p_0}{p} - \frac{p}{p_0}\right)}{\left(\frac{p}{p_0} + \frac{p_0}{p}\right)} \tag{8}$$

It is important to observe that no preferred frame exists in any relativistic analysis. We also can choose the center of mass of both masses as reference frame, in which, we need to take in consideration for calculations m and M, as simultaneously variables, introducing additional complications to the analysis. Or taking m fixed and M moving curvilinear around m. In our opinion, it is logic defining M as a massive object to take it as fixed for this development.

The corresponding Force between them will become:

$$\mathcal{F}_C = m.\mathcal{G}_C = \frac{\dfrac{2.G.M}{r^2}.\dfrac{p}{V_0} - p.\dfrac{dv}{dr}.\left(\dfrac{p_0}{p} - \dfrac{p}{p_0}\right)}{\left(\dfrac{p}{p_0} + \dfrac{p_0}{p}\right)} \tag{9}$$

The values of velocity v and linear momentum p of moving mass m with the sub-index zero, V_0 and p_0, indicate their values at the closest position of m to M, in which the value of radius becomes r_0 and in where, along all this movement of m around the fixed mass M, angular momentum is preserved constant: $m.\omega^2.r = Const.$

IV. GRAVITATION FOR RECILINEAR MOTION IN VECTORIAL RELATIVITY.

The analysis of rectilinear motion of a moving mass m attracted by a mass M, on the line joining both centers of mass, has been the most studied case since Galileo in 1638 obtained his law of falling bodies ($h = \frac{1}{2}g.t^2$) and since Newton in 1687 completed his three laws of motion (**1.** "***An object will remain at rest or in uniform motion in a straight line unless acted upon by an external force***". **2.** "***The rate of change of momentum (m.v) of a body is proportional to the resultant force acting on the body and is in the same direction".*** **3.** "***Any force is opposed by another equal and opposite***") combined with his law of Universal Gravitation ("***any two objects attract each other with a force that acts in the line joining them, such that its intensity varies directly as the product of their masses and inversely as the square of the distance between their centers of mass, times a constant of proportionality G***").

In fact, the definition of weight of an object is precisely the force of attraction exerted by earth on such object measured statically, but in free fall motion the attraction of earth makes the object to displace along the line joining object's center of mass with earth's center of mass.

We have seen in [3] that the Newton's original gravitation field definition, $\mathcal{G} = \frac{G.M}{r^2}$, created by a massive mass M when applied to a moving variable mass m in curvilinear motion was forced to change to expression (8). In the rectilinear case, Cavendish's experiments in 1798 gave as result the first measuring of the constant of gravitation, G, 111 years after Newton published all his laws of motion and gravitation. All the last measurements until today corroborate his obtained value, see for example the recent publication, *Science 5, January 2007, Vol. 315 No. 5808, pp 74 -77*: "Here, we report a value of G = 6.693 x 10^{-11} cubic meters per kilogram second squared, with a standard error of the mean of ±0.027 x 10^{-11} and a systematic error of ±0.021 x 10^{-11} cubic meters per kilogram second squared". It can be indirectly inferred (because his results were referred to earth density) that Cavendish results about measuring this constant differs only 1% with the 2007 value, in a dynamic measurement over an oscillation-torsion of two masses of 15 g, attracted by to masses of 1.5 Kg. with approximately a ten minutes-period of oscillation.

These results indicate positively that the original Newton's Law of Universal Gravitation holds for rectilinear (or radial) attraction. In Vectorial Relativity we will state as an exact result that of Newton's Law of Universal Gravitation only for the case of rectilinear and radial displacement of the moving mass. It can be observed that in this motion angular momentum has always a null value $m.\omega^2.r = 0$, because angular velocity has always a null value, $\omega = 0$. So, angular momentum, also in this case, maintains its value, namely it preserves constant its value, which is consistent within an isolated system. In radial attraction, the closest distance between m and M has no any meaning, thus it is also a consistent result that extremes values of velocity and momentum of moving mass m with the sub-index 0, V_0 and p_0, do not appear in the expression of original Newton's Law of Universal Gravitation. Also, as a consistent result, if V_0 and p_0 are substituted by v, p, respectively, in equations (8) and (9) of curvilinear motion, we obtain the correspondent radial equations for rectilinear motion or Newton's original gravitational law. These observations, of course, support our previous assumption that the original Newton's Law of Universal Gravitation is correct for the case of radial displacement of the moving mass m attracted by the massive mass M. So, the force for a rectilinear, or radial, displacement of a moving mass attracted by a fixed mass M, as Newton originally established, is given by:

$$\mathcal{F}_R = \mathcal{G}_R .m = \frac{G.M.m}{r^2} \qquad \Leftrightarrow \qquad \mathcal{G}_R = \frac{G.M}{r^2} \tag{10}$$

Discussion: As we have observed, the original Newton's Law of Universal Gravitation gets a modified expression when it is referred to curvilinear motion. In this new presentation it does not only depend on the masses and the distance between their centers of mass but also on the linear momentum, total acceleration, radial acceleration and the velocity of the moving mass as indicated in (9). Also, we have observed that this law reduces to its original presentation when it is referred to radial and rectilinear motion

V. INERTIAL AND GRAVITATIONAL MASSES. EQUIVALENCE PRINCIPLE.

This aspect is accessed here in order to put clear, in author's opinion, the concepts involved in this antique controversy, and how to remove it.

The reason for appearing these two concepts of mass is because there are two force concepts involved in this discussion (we are going to use our new definitions):

1) That of the Second Newton's Law, $\mathcal{F} = \frac{d}{dt}(m.v)$ and

2) That of the Newton's Law of Universal Gravitation $\mathcal{F} = m.\mathcal{G}$, where $\mathcal{G}$ is the gravitational field which has a different shape compared to that coined by Newton, for curvilinear motion [3].

In the first case, mass as a measure of the quantity of material contained in an object, is referred to the resistance to a change of motion when the object is under the effect of an applied force, and it is referred to, in the physics jerk to the inertial mass. In the second case, *with the same meaning of measure previously used,* gravitational mass m of this same object is its resistance to motion induced by a gravitational field of force $\mathcal{G}$ produced by another mass M. Obviously both masses are the same mass, the same measure of quantity of material contained in such object. And as we observe, each force causes in the mass m a different effect. The particular case of a static measurement will take us to measure the mass at rest, $\overset{0}{m}$. We can imagine this situation by doing an experiment similar to that of Einstein's principle of equivalence in where it is supposed an observer inside an isolated box located over an automatic balance that gives him the lecture of the weight of all system.(balance + observer + display inside the box). In principle, let the system at rest on earth to indicate a weight of **980 Newton**. This is what display will inform to the observer. Given that at the local surface of earth measured magnitude of the gravitational field is $\mathcal{G}=9.8\,\frac{m}{sec^2}$, then mass at rest for this example becomes $\overset{0}{m}=100\,Kg$.

Now, let the same system be located at a point P deep in the space without the influence of gravitation forces and a second observer is located outside the box and at the same point P, with his own equipment for measurements. Obviously, due to the system is floating in the space, display inside the box will read **0 Newton**. Suddenly, a constant force $\mathcal{F}$ is exerted at the bottom of the system that accelerates it, such that display inside the box, by virtue of the third Newton's Law, reads again 980 Newton. Observer inside the box would think that he is at earth again under the effect of gravity, because he is at rest relative to the box and that he statically is viewing at the display a weight of 980 Newton, and similarly, system mass has not changed, and it has a constant value $\overset{0}{m}=100\,Kg$. Also, he feels a supposed equivalent gravitational field of $\mathcal{G}=9.8\,\frac{m}{sec^2}$. In sum, he cannot distinguish between the applied force $\mathcal{F}$ in this second case from weight given by the gravitational force of earth in the first case.

On the other hand, second observer measures a constant force of 980 Newton applied at the bottom on the system. So, application of Newton's Second Law gives him: $980=\frac{d}{dt}(m.v)=m.\frac{dv}{dt}+v\frac{dm}{dt}=m.a+v\frac{dm}{dt}$, such that variation in speed and variation in mass have to compensate between them in order to maintain force constant. Relativistic mass at any speed v, according to Vectorial Relativity Theory has the expression:

$$m=\frac{\overset{0}{m}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \qquad \text{(5) bis}$$

The previous analysis is simplified given that force is constant and we know the general expression of the variation of kinetic energy [2] from a point r_0 until a distance $r-r_0$, for $r_0, V_0=0$.

$$K=\mathcal{F}.r=m.\left(2.v^2-c^2\right)+\overset{0}{m}.c^2$$

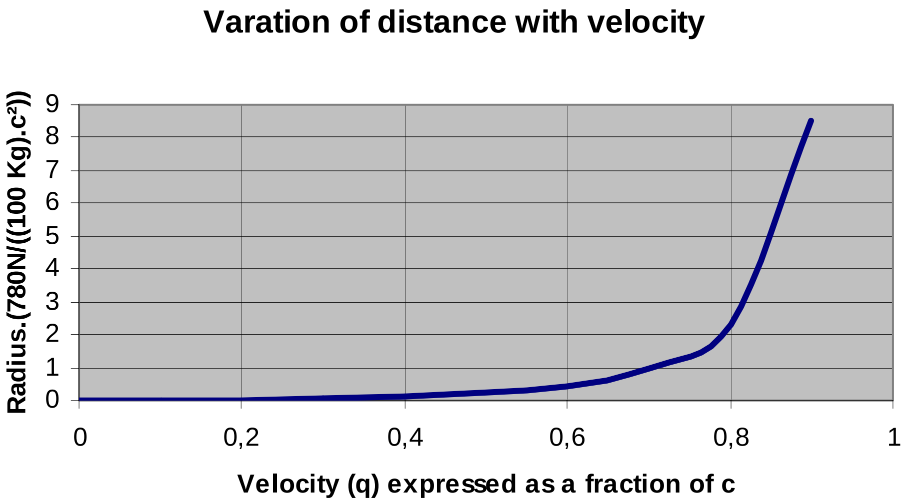
So, we can have the value of distance r in function of speed v, for a constant applied force $\mathcal{F}$:

$$.r=\overset{0}{m}.\frac{\dfrac{(2.v^{2}-c^{2})}{\left(1-\dfrac{v^{2}}{c^{2}}\right)^{\frac{3}{2}}}+c^{2}}{\mathcal{F}} \qquad (11)$$

By doing the velocity of the system a function proportional to light speed through a constant q, $v=q.c$, for q varying as $0\leq q<1$, we obtain for the radius times a constant factor:

$$r.\left[\frac{\mathcal{F}}{\overset{0}{m}.c^{2}}\right]=\left(\frac{2.q^{2}-1}{\left(1-q^{2}\right)^{\frac{3}{2}}}+1\right) \qquad (12)$$

Whose interdependency can be represented as:



In the previous example of an accelerated variable mass m, measured by the second observer, who is not part of the system, he is only observing from outside what is indeed occurring there. It is inferred from (5) that the measured mass m is continuously increasing because the speed is always increasing, although for first observer, he feels he is at rest and observes the display indicating that system has a constant weight of 780 Newton and “of course” a mass at rest of 100 Kg. On the other hand, when in the first part of this imaginary experiment the system was located fixed relative to earth, and a second observer was located also fixed relative to earth, both observers measured the mass at rest, $\overset{0}{m}$ as having a value of 100 Kg. Obviously, first observer cannot differentiate the initial state from the last one, and for him, in these two conditions he feels the same motionless situation. For second observer both situations are completely different: In the first case, he measured a mass at rest, stationary, on earth, with a constant value $\overset{0}{m}$. Contrarily, measurement in second case of mass m is a dynamic process because mass is continuously being increased, and measured mass m is not equal to that of the first case. We can assert that from a relativistic point of view, if you locate a local observer to analyze a determined event, you must take another observer located in a different inertial system analyzing the same event to conclude something. If your conclusion, supposedly relativistic, takes in consideration only one observer, relativistic speaking, you are going to get in trouble in some moment.

The last thing that is relevant to observe is that resemblance between weight of a mass $\overset{0}{m}$ at a radius r_1 on earth and a constant force $\mathcal{F}_1$ accelerating a system has not a happy end: For the same mass, if radius change, weight of mass $\overset{0}{m}$ changes and applied force needs to change Thus, dynamically these two concepts are not the same thing.

It is worthwhile to mention also that **equivalence principle** in the assertion that the gravitational "force" as experienced locally by one observer while standing on a massive body (such as Earth) can be resembled the same as the *force* experienced by the same observer in an accelerated frame of reference, as it was explained in the previous example, is called Einstein's equivalence principle due to Einstein proposed it in 1907. For establishing it, Einstein used two inertial systems K and K', but only take into account one observer.

Another known **equivalence principle,** experimentally demonstrated by Galileo in 1638 and exposed as the law of falling bodies ('Discourses and Mathematical Demonstrations Concerning Two New Sciences', Leiden, 1638)**:** All objects fall the same way under the influence of a gravitational field (free fall). The following can be commented from this equivalence principle:

1. The force exerted by earth on objects is independent of type of constituent material of such objects and depends only (weight) on the quantity of matter (mass). In this sense, the magnitude of the gravitational force attracting a mass located at a point P in space is proportional to attracted mass, and such proportionality is independent of constituent matter.

2 An observer within a box in free fall experiences a weightlessness sensation, as he was floating. He would feel this same sensation if he were in the empty space in a fixed point without the influence of gravitation. Moreover, this same sensation is felt by the observer if he were in a uniform rectilinear motion.

3 If the box with an observer inside were moving like a satellite in an elliptic orbit at constant angular momentum around earth, where velocity is not constant, he also would experience a weightlessness sensation, as in vertical free fall. Under these characteristics all the following environments: observer at rest, in a uniform rectilinear motion, in a free fall rectilinear or curvilinear under the influence of a gravitational field, can be considered to be defined as inertial frames, despite that gravitation can produce uniform (vertical free fall) or non-uniform (curvilinear) accelerated motion.

So, *inside the box an observer locally could not distinguish among the different inertial frames previously described.*

It can be shown from a vectorial relativistic point of view the experimental observation that "All objects fall the same way under the influence of a gravitational field" is a theoretical consequence of the definition of Gravitational Radial Force, as the product of the attracted mass by the gravitational field, because gravitational field can be put as independent of mass and depending only on kinematical variables. **Thus, mass is defined as unique and the same either that considered in the second Newton's Law or that considered in Gravitation.** ***In other words, Gravitation is***

positively a force as Newton stated previously, but with the different expressions corrected by Vectorial Relativity, and emphasizing that some equivalence principles, as the last one, can be considered relevant only to define inertial frames of reference.

In fact, by equaling the gravitational radial force exerted by mass M and force definition contained in Newton's second law for considering the resulting ***rectilinear*** motion of m in free fall, we have:

$$\mathcal{F}_R = m.\mathcal{G}_R = \frac{d(m.v)}{dt} = m.\frac{dv}{dt} + v.\frac{dm}{dt} \tag{13}$$

For $m = \dfrac{\overset{0}{m}}{\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{3}{2}}}$ (14)

$$\Rightarrow \qquad dm = \frac{3.m.v.dv}{\left(c^2 - v^2\right)}, \tag{15}$$

Substituting:

$\mathcal{F}_R = m.\mathcal{G}_R = m.\dfrac{dv}{dt} + v.\dfrac{3.m.v}{\left(c^2 - v^2\right)}.\dfrac{dv}{dt} = m.\dfrac{dv}{dt}.\left(1 + \dfrac{3.v^2}{\left(c^2 - v^2\right)}\right) = m.\dfrac{dv}{dt}.\dfrac{c^2 + 2.v^2}{c^2 - v^2}$, and finally we obtain:

$$\mathcal{G}_R = \left(\frac{c^2 + 2.v^2}{c^2 - v^2}\right).\frac{dv}{dt} \tag{16}$$

In the previous equation (16) it can be seen that the magnitude of Gravitational Field is different from the acceleration magnitude. Also it can be seen that gravitational field magnitude is independent of mass magnitude, and thus, as a pure kinematical property applicable to any mass in vertical free fall it can be concluded: ***Any mass will develop the same acceleration and speed in vertical free fall***. Result expressed in equation (16) indicates that gravitational field in rectilinear motion is not only conceptually different but also it is numerically greater than acceleration. A similar result is possible to be found for curvilinear motion, but interpretation is not the same as for vertical free fall: In curvilinear motion it can be observed that there exists equilibrium between the Gravitational force and centrifugal force (summation of both forces is zero). Gravitational is a force with radial direction and sense towards the center of mass of the fixed mass M and Centrifugal Force is perpendicular to the tangential velocity of moving mass m, and its radial component is opposite to the gravitational attraction. The signs involved are taking into account inside next equations:

$$-\vec{\mathcal{F}}_C = -m.\vec{\mathcal{G}}_C = \frac{d(m.\mathbf{v})}{dt} = m.\frac{d\mathbf{v}}{dt} + \mathbf{v}.\frac{dm}{dt} = m.\frac{d\mathbf{v}}{dt} + \mathbf{v}.\frac{3.m.v}{\left(c^2 - v^2\right)}.\frac{dv}{dt} \tag{17}$$

Given the following vector definitions [5]:

$$\mathbf{r} = r.\mathbf{U}_r \quad \Rightarrow \quad \mathbf{v} = \frac{dr}{dt}.\mathbf{U}_r + r.\omega.\mathbf{U}_\theta \quad \Rightarrow \quad \frac{d\mathbf{v}}{dt} = \left[\frac{d^2r}{dt^2} - r.\left(\frac{d\theta}{dt}\right)^2\right].\mathbf{U}_r + \left[r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}.\frac{d\theta}{dt}\right].\mathbf{U}_\theta$$

Substituting properly in (17), we obtain

$$\left[m.\frac{d^2r}{dt^2}-r.m.\left(\frac{d\theta}{dt}\right)^2+\frac{dm}{dt}.\frac{dr}{dt}\right].\mathbf{U}_r+\left[r.m.\frac{d^2\theta}{dt^2}+2.m.\frac{dr}{dt}.\frac{d\theta}{dt}+r.\frac{dm}{dt}.\frac{d\theta}{dt}\right].\mathbf{U}_\theta=-\mathcal{F}.\mathbf{U}_r=-m.\mathcal{G}_C.\mathbf{U}_r$$

From here we have two scalar equations. That of the expression multiplying the unit vector $\mathbf{U}_\theta$, which should be equaled to zero, leads to achieve the angular conservation law, $\omega.r^2.m=\text{constant}$:

$$r.m.\frac{d\omega}{dt}+2.m.\frac{dr}{dt}.\omega+r.\frac{dm}{dt}.\omega=0\Rightarrow\frac{d\omega}{\omega}=\left(2.\frac{dr}{r}+\frac{dm}{m}\right)\Rightarrow\mathbf{ln}\frac{\omega}{\omega_0}=-\mathbf{ln}\left(\frac{r^2}{r_0^2}.\frac{m}{m_0}\right)\Rightarrow\omega.r^2.m=\omega_0.r_0^2.m_0$$

And the first one, equaled to $-m.\mathcal{G}_C$, gives us the desired Gravitational Field's expression:

$$-\mathcal{G}=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2+\frac{dm}{m.dt}.\frac{dr}{dt}=\frac{d^2r}{dt^2}-r\omega^2+\frac{3.v}{c^2-v^2}.\frac{dv}{dt}.\frac{dr}{dt}$$

$$-\mathcal{G}=\frac{d^2r}{dt^2}-r\omega^2+\frac{3.v}{c^2-v^2}.\frac{dv}{dt}.\frac{dr}{dt} \qquad (18)$$

It is noticed that the Gravitational Field in (18) also depends on radius, which in turn depends on mass, according to the angular conservation law, previously developed through the second scalar equation. So, we must expect that "***different masses have distinct trajectories***".

Another observation detected in (18) is that although Gravitational Field has dimension of acceleration, $\left[\frac{m}{sec^2}\right]$, they are, as before for rectilinear free fall, numerically and conceptually different. The only case in which Gravitational Field would be numerically equal to acceleration is for constant mass m, as it can be established from expression (13). But, from the relativistic expression (14) constant mass would be possible only if its velocity is constant and this means constant radius or circular motion. In fact, from (18) for constant velocity v, constant radius r and angular velocity ω, the value of gravitational field equals the centripetal acceleration, $\mathcal{G}=a=\omega^2.r$. It is noteworthy that also for this case, different masses originate different radiuses and circles.

As we observe from this work, while in Newtonian mechanics mass is an intrinsic property of a body, in a relativistic environment mass also depends, relative to some reference, on the relationship between its speed and the universal constant speed of light c. As we have remarked before Newtonian mechanics has slight discrepancies between its predictions and actual events, as the motions of the planet Mercury or being strictly rigorous, the non-prediction of planet's precession. This last fact has given more strength to a relativistic conception of nature, because it accounts for it.

But, why is so important the analysis of the equivalence principle?. At first, during XVIII and XIX centuries, it was the study of the universal law of gravitation for obtaining the experimental value of the gravitation constant G, very difficult to measure, until Henry Cavendish in 1798 did his famous experiment, which with some refinements is used until today for measuring this constant. In XX century, the equivalence principle got a major importance because according to Einstein if inside a box an observer locally could not distinguish among the different inertial frames, specially into the idea of moving in rectilinear uniform motion, which can be rapidly related to the shortest distance between two points, and if this is so, also the case of moving following an elliptical path in free fall under a gravitational field, which can be related to the shortest path between two points in a curved spatial surface (geodesic): Then, if Gravitation makes an object in free fall take a curvilinear path it must be because Gravitation modifies the curvature of the space-time, as it was called after Einstein proposed its Special Theory of Relativity (SRT). This kind of reasoning, led Einstein to propose the General Theory of Relativity, reducing gravitation only to a geometric characteristic of the space-time and at the non-consideration of gravitation as a force. All this century and still, scientists have been discussing relativistic Einstein ideas.

However, in author's opinion, Einstein's geometric view of gravitation could be a valid way to face the problem of motion of masses under Gravitation. But, Newton's consideration of gravitation as a force, and the subsequent derivation of gravitation field definition, has worked wonderfully in equations until the problem of precession of Mercury appeared. After SRT was proposed and "accepted" by the scientific community, concepts of mass, length and time changed but preserving its identity in Physics and they began to be used as we were used to. With Gravitation we can do the same by preserving its identity as a force. In fact, in this work we have done so. We are proposing a modified expression of Gravitational Force as that of equation (9) in order to make it compatible with the law of conservation of Angular Momentum, and we have obtained that when the new expression of the Gravitation Force (9) is used in dynamic physics problems, precession is considered in their results [4] [5].

VI. CONCLUSIONS

The accuracy of our new definitions of mass, energy or Gravitational Force obtained in this work and in previous ones, referred in the Reviews published in this and previous issues, will probably require further research and complex experiments with known rest masses accelerated at speeds close to that of light in order to establish the correctness of our work. These tasks probably could be possible to achieve by next years.

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