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Precession in Vectorial Relativity

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ABSTRACT: In previous work it was demonstrated that the correct equation that governs gravitational effect of a massive body, considered it with a fixed and constant mass, on a photon moving at velocity c and momentum p, which has a variable mass in its curved path given by $m=\frac{p}{c}$, became: $\frac{d^2r}{dt^2}-r.\omega^2+\frac{\mathrm{G}}{c^2}\omega^2.r^2=0$, where the value of the gravitational field G exerted by the massive body on the variable mass of photon was found to be: $\mathrm{G}=\frac{\frac{2.G.M}{r^2}}{\frac{p}{p_0}+\frac{p_0}{p}}$, denoting by p_0 the constant value of the linear momentum for a photon attracted by the massive body, at its nearest point (r_0) and p a generic value of the linear momentum of the photon at any other point (r). One of the reasons of the success of Einstein's General Theory of Relativity (GTR) was that it allowed to calculate planet's precession (rotation of the elliptical path axis with time, i.e.: Mercury Precession). This fact, that its occurrence has been experimentally observed, is not accounted by classic Kepler's Laws or Newton Laws because it is only applicable to constant masses. In this work, as a direct result of only considering mass as variable inside equations and applying known and accepted physical laws, it is shown that precession appears as one of the natural outcomes, preserving speed of light constant.

KEYWORDS: Universal Gravitation, Kepler Laws, Vectorial Relativity, STR, GTR.

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I. INTRODUCTION

This work is a continuation of previous one on Gravitation [2], also reviewed in this issue [3]. In the present work is obtained an approximate expression of angle of precession as much in photons as in planets. In principle, in order to put clear the used procedure for achieving such approximation we repeat that one used in the deduced relationship between angle and radius for constant masses in section II. In Sections III and IV is discussed about a way to show that consideration of mass as variable allows predicting effect of precession observed in planets motion. An approximate way of calculating precession angle is presented and opened this theme for further research to obtain an exact solution or a better approximation of precession calculation. At the end of this work are presented some conclusions.

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II. *CLASSIC EXPRESSION OF RADIUS IN FUNCTION OF ANGLE.*

In previous work, for constant masses it was presented how Kepler's Laws can be directly deduced from Newton's Universal Law of Gravitation [2], let's resume it here. Letting $\mathbf{r} = r.\mathbf{U_r}$, the following expressions of velocity and acceleration can be obtained:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}.\mathbf{U_r} + r.\frac{d\theta}{dt}.\mathbf{U}_\theta \quad \text{and} \quad \mathbf{a} = \frac{d\mathbf{r}}{dt} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right].\mathbf{U_r} + \left[r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}\frac{d\theta}{dt}\right].\mathbf{U}_\theta \qquad (1)$$

By applying the definition of Force and Newton's Universal Gravitation Law to two attracting bodies of constant masses M and m, we can write that:

$$\frac{d\mathbf{p}}{dt} = m.\mathbf{a} = -F.\mathbf{U_r} = -\frac{G.M.m}{r^2}.\mathbf{U_r} \qquad \Rightarrow \qquad \mathbf{a} = -\frac{G.M}{r^2}.\mathbf{U_r} \qquad (2)$$

Substituting the obtained expression (1) of acceleration in (2) we arrive at:

$$\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right].\mathbf{U_r} + \left[r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}\frac{d\theta}{dt}\right].\mathbf{U}_\theta = -\frac{G.M}{r^2}\mathbf{U_r} \qquad (3)$$

This vectorial equation creates the following two scalar equations:

$$\begin{aligned} &1) \quad r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}\frac{d\theta}{dt} = 0 \\ &2) \quad \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{G.M}{r^2} \end{aligned} \qquad (4)$$

By remembering that angular velocity is defined as $\omega = \frac{d\theta}{dt}$, first equation generates Angular Momentum Conservation Law:

$$r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}\frac{d\theta}{dt} = 0 \Rightarrow \frac{\frac{d^2\theta}{dt^2}}{\frac{d\theta}{dt}} + 2.\frac{\frac{dr}{dt}}{r} = \frac{\frac{d\omega}{dt}}{\omega} + 2.\frac{\frac{dr}{dt}}{r} = 0 \Rightarrow \frac{d\omega}{\omega} + 2.\frac{dr}{r} = 0 \Rightarrow Log\frac{\omega}{\omega_0} = Log\frac{r_0^2}{r^2}$$

$$\omega.r^2 = \omega_0.r_0^2 = \text{Constant} \quad \Rightarrow \quad \frac{d\omega}{dt}.r^2 = \text{Constant} \qquad \text{for } \omega_0,\ r_0 \text{ measured at perihelion} \qquad (5)$$

Development of the second scalar equation allows obtaining the following expression:

$$q^2 = \left(\frac{dr}{dt}\right)^2 = r_0^4.\omega_0^2.\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) - 2.G.M.\left(\frac{1}{r_0} - \frac{1}{r^{\cdot}}\right) = r_0^2.v_0^2.\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) - 2.G.M.\left(\frac{1}{r_0} - \frac{1}{r^{\cdot}}\right) \qquad (6)$$

Doing $\frac{d\theta}{dt}.r^2 = \frac{d\theta}{dr}.\frac{dr}{dt}.r^2 = \frac{d\theta}{dr}.q.r^2 = \mathbf{K} \quad \Rightarrow \quad d\theta = \frac{K.dr}{q.r^2}$ (7)

Defining $u = \frac{1}{r} \Rightarrow du = -\frac{dr}{r^2}$, and substituting q by its equivalent expression, we obtain:

$$d\theta = \frac{-du}{\sqrt{\left(u_0^2 - u^2\right) - \frac{2.G.M}{r_0^2.v_0^2}.\left(u_0 - u\right)}} = \frac{-du}{\sqrt{\left(u_0 - h\right) - \left(u - h\right)}} \quad \text{for} \quad h = \frac{G.M}{r_0^2.v_0^2}$$

By integrating both sides of this equation we get the final solution for angle:

$$\theta = arc\cos\left(\frac{\frac{1}{r} - h}{\frac{1}{r_0} - h}\right) \quad \Leftrightarrow \quad r = \frac{\frac{1}{h}}{1 + \left(\frac{1}{r_0.h} - 1\right).\cos\theta} \quad \Leftrightarrow \quad r = \frac{1/h}{1 + e.\cos\theta} \tag{8}$$

Equation (8) represents a conic with eccentricity $e = \frac{1}{r_0.h} - 1$, with one focus at origin. Applied to planets, for $\theta = 0 \Rightarrow r = r_0$, is the value of radius at perihelion.

However, it is necessary to introduce the following important comment: because equation (8) represents a perfect conic, the phenomenon of precession (rotation of the conic plane axis, respect to the center of the conic) is absent in this solution. The nonappearance of the phenomenon of the precession in this analysis, in author's opinion, is given by the non-consideration of the relativistic dependence of mass on its speed and on the speed of light. Later, inside the next discussion on photon's and planet's precession we will recall expressions, similar to these worked here.

III. PHOTON'S PRECESSION

Although, it is difficult to test its motion, we only take example of photon as a pedagogic way to attack the general problem of determining photon's movement, including that of precession. We are going to follow a similar analysis to that done in previous Section.

We have introduced in the abstract, that photon, always moving at a constant speed c with a linear momentum p, has a mass given by $m = \frac{p}{c}$ [1]. But, It is noteworthy that photon, whose speed is constant, has variable mass under a gravitational field, either in curvilinear or in rectilinear motion.

In previous work we have discussed photon's motion [2], and so do we in this current issue, in [3], in where we arrived at two equations that govern photon's movement (first of them gives us the Angular Momentum Conservation Law). They are:

1) $$r.\frac{d^2\theta}{dt^2}+2.\frac{dr}{dt}.\frac{d\theta}{dt}+\frac{r}{p}.\frac{dp}{dt}.\frac{d\theta}{dt}=0 \tag{9}$$

2) $$\frac{d^2r}{dt^2}-r.\omega^2+\frac{\mathrm{G}}{c^2}\omega^2.r^2=0 \tag{10}$$

Where, Gravitational Field G produced by a constant mass M, acting on a variable photon's mass m, exhibits the following new expression:

$$\mathrm{G}=\frac{\dfrac{2.G.M}{r^2}}{\dfrac{p}{p_0}+\dfrac{p_0}{p}} \tag{11}$$

Namely, gravitational field G, produced by a massive body of mass M acting on photon, not only depends on mass M and radius r, as it was originally established by Isaac Newton, but also on the current value of linear momentum p and on its value at perihelion, p_0. In addition to equation (9), we also have found in [2] that photon's radial velocity $q=\frac{dr}{dt}$ became:

$$q^2=\frac{\omega_0^2.r_0^4.p_0^2}{p^2}.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)-2.G.M.\frac{p_0}{p}.\left(\frac{1}{r_0}-\frac{1}{r}\right)=\omega^2.r^4.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)-2.G.M.\frac{p_0}{p}.\left(\frac{1}{r_0}-\frac{1}{r}\right) \tag{12}$$

and that the expression of linear momentum was given by:

$$\begin{aligned} p&=p_0.\left[\sqrt{\left(\frac{G.M}{c^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right)^2+1}-\frac{G.M}{c^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right] \\ p_0&=p.\left[\sqrt{\left(\frac{G.M}{c^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right)^2+1}+\frac{G.M}{c^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right] \end{aligned} \tag{13}$$

With these tools in hand we will try to obtain the relationship between radius r and angle θ, which will allow us calculating photon's precession. From the first equation of motion displayed in (9), it is obtained something equivalent to the conservation of angular momentum:

$$\Rightarrow \ \mathbf{ln}\frac{\omega}{\omega_0}=-2.\mathbf{ln}\frac{r}{r_0}-\mathbf{ln}\frac{p}{p_0} \quad \Rightarrow \quad \omega.r^2.p=\omega_0.r_0^2.p_0=K=\text{Constant} \tag{14}$$

Starting from (14) and using $\omega=\frac{d\theta}{dt}=\frac{d\theta}{dr}\frac{dr}{dt}=\frac{d\theta}{dr}.q$, we have:

$$\omega.r^2.p=\frac{d\theta}{dr}.q.r^2.p=K \quad \Rightarrow \quad d\theta=\frac{K.dr}{q.r^2.p} \tag{15}$$

$$d\theta = \frac{K}{\sqrt{\frac{K^2}{p^2}.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)-2.G.M.\left(\frac{1}{r_0}-\frac{1}{r}\right).\frac{p_0}{p}.r^2.p}}.dr = \frac{dr}{\sqrt{\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)-\frac{2.G.M}{K^2}\,p_0.p.\left(\frac{1}{r_0}-\frac{1}{r}\right).r^2}}$$

By doing: $u=\frac{1}{r} \Rightarrow du=-\frac{dr}{r^2};\ h=\frac{G.M.p_0^2}{K^2};\ p=p_0.\left[\sqrt{\left(\frac{G.M}{c^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right)^2+1}-\frac{G.M}{c^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right];$

And for $p=p_0.f\left(1/r\right) \Rightarrow p=p_0.f(u)$, it can take the following expressions:

$$d\theta=\frac{-du}{\sqrt{.\left(u_0^2-u^2\right)-2.h.\left(u_0-u\right).f(u)}} \quad \text{for } f(u)=\sqrt{\left(\frac{G.M}{c^2}.\left(u_0-u\right)\right)^2+1}-\frac{G.M}{c^2}.\left(u_0-u\right) \tag{16}$$

$$d\theta=\frac{-du}{\sqrt{.\left(u_0-h.f(u)\right)^2-\left(u-h.f(u)\right)^2}} \tag{17}$$

Differential expressions (16) and (17) do not have a known solution, but it is possible to work on them in an approximate configuration to demonstrate that in a complete “oval” revolution, after taking again radius r_0, as it was indicated by equations to preserve the constant angular momentum law, swept angle is different of 2π.

A. Approximation to the integration of angle.

We know that the differential equation in (17), for a constant value of $f(u)$**,** has a known solution:

$$\int_{r_0}^{r_a} d\theta=\int_{r_0}^{r_a}\frac{-du}{\sqrt{.\left(u_0-h.f(u)\right)^2-\left(u-h.f(u)\right)^2}}=arc\cos\left.\left(\frac{\frac{1}{r}-h.f(a)}{\frac{1}{r_0}-h.f(a)}\right)\right|_{r_0}^{r}=\theta+C \tag{18}$$

This integration between perihelion and aphelion in (18), will be our starting point for estimating an approximate value of equation (17), Where a is some constant and C is a constant of integration, that is null because for $\theta=0,\ u=u_0$. Thus, $\theta=0=arc\cos(1)=0+C=0 \quad \Rightarrow \quad C=0$.

As it can be observed from the graph at the end of this work $f(u)$ can be represented by a triangle-rectangle with the following sides: the basis, $\frac{G.M}{c^2}.\left(u_0-u\right)$**,** and the height, with a constant value of unity. In this way hypotenuse becomes, $\sqrt{\left(\frac{G.M}{c^2}.\left(u_0-u\right)\right)^2+1}$. By subtracting from the hypotenuse the basis, we obtain $f(u)$, which is that part of the hypotenuse not black bolded. For doing the integration the evaluation of $f(u)$ is done at the middle of the interval and it is represented by the bolded green line. From here we can observe that $f(u) \leq 1$.

By doing the integration for example, between perihelion and aphelion, by short intervals where the value of $f(u)$ is evaluated constantly at the middle of each interval, and adding all these partial integrations, we can then have an approximate of the total integration.

We can observe that $f(u) \leq 1$ for any value of u, namely, for $r = r_0 \Rightarrow f(u) = 1$ and for any other value of $r > r_0 \Rightarrow f(u) < 1$. In order to see what we are going to do, let r_0 *and* r_a, be the extremes of integration in oval motion, N be the number of intervals and $\Delta = \dfrac{\frac{G.M}{c^2}.\left(\frac{1}{r_0} - \frac{1}{r_a}\right)}{N}$ be interval amplitude. In this way, we can say that r_1 is the end of one Δ, namely $1.\Delta$ is determined by the extremes r_0 and r_1; two Δ determines also two extremes, r_0 and r_2; and in general $i.\Delta$, i the interval ordinal, determines r_0 and r_i. From this, we have that $i\Delta = i\dfrac{\frac{G.M}{c^2}.\left(\frac{1}{r_0} - \frac{1}{r_a}\right)}{N} = \dfrac{G.M}{c^2}.\left(\dfrac{1}{r_0} - \dfrac{1}{r_i}\right)$, and r_i becomes $r_i = \dfrac{r_0}{1 - i\frac{1 - r_0/r_a}{N}}$. Evaluation of $f(u)$ should be, as an evident criterion, at the middle point of the last interval integrated, $r_{01} = \dfrac{r_0}{1 - (i - 1/2)\frac{1 - r_0/r_a}{N}}$, whose interval extremes are:

$$\left[r_{i-1} = \frac{r_0}{1 - (i-1)\frac{1 - r_0/r_a}{N}},\ r_i = \frac{r_0}{1 - i\frac{1 - r_0/r_a}{N}} \right].$$

Let's see examples of the integration in each interval:

<u>INTEGRATION BETWEEN:</u> <u>EVALUATED $f(r)$ AT:</u> <u>RESULTANT ANGLE θ:</u>

1) $\begin{bmatrix} r_0 \\ r_1 = \dfrac{r_0}{1 - \frac{1 - r_0/r_a}{N}} \end{bmatrix}$ $\quad r_{01} = \dfrac{r_0}{1 - (1 - 1/2)\frac{1 - r_0/r_a}{N}}$ $\quad arc\cos\left(\dfrac{\frac{1}{r_1} - h.f(r_{01})}{\frac{1}{r_0} - h.f(r_{01})} \right) - arc\cos\left(\dfrac{\frac{1}{r_0} - h.f(r_{01})}{\frac{1}{r_0} - h.f(r_{01})} \right)$

2) $\begin{bmatrix} r_1 = \dfrac{r_0}{1-1\dfrac{1-r_0/r_a}{N}} \\ r_2 = \dfrac{r_0}{1-2\dfrac{1-r_0/r_a}{N}} \end{bmatrix}$ $r_{12} = \dfrac{r_0}{1-(2-1/2)\dfrac{1-r_0/r_a}{N}}$ $arc\cos\left(\dfrac{\dfrac{1}{r_2}-h.f(r_{12})}{\dfrac{1}{r_0}-h.f(r_{12})}\right) - arc\cos\left(\dfrac{\dfrac{1}{r_1}-h.f(r_{12})}{\dfrac{1}{r_0}-h.f(r_{12})}\right)$

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i) $\begin{bmatrix} r_{i-1} = \dfrac{r_0}{1-(i-1)\dfrac{1-r_0/r_a}{N}} \\ r_i = \dfrac{r_0}{1-i\dfrac{1-r_0/r_a}{N}} \end{bmatrix}$ $r_{i-1,i} = \dfrac{r_0}{1-(i-1/2)\dfrac{1-r_0/r_a}{N}}$ $arc\cos\left(\dfrac{\dfrac{1}{r_i}-h.f(r_{i-1,i})}{\dfrac{1}{r_0}-h.f(r_{i-1,i})}\right) - arc\cos\left(\dfrac{\dfrac{1}{r_{i-1}}-h.f(r_{i-1,i})}{\dfrac{1}{r_0}-h.f(r_{i-1,i})}\right)$

In sum, the total integration for elliptical case, for N intervals, in where r_N, the last extreme of integration equals radius at aphelion, r_a, half cycle, can be achieved by adding all partial results:

$$\theta = \int_{r_0}^{r_a} \frac{-du}{\sqrt{.(u_0 - h.f(u))^2 - (u - h.f(u))^2}} = \sum_{i=1}^{N}\left\{ arc\cos\left(\frac{\dfrac{1}{r_i}-h.f(r_{i-1,i})}{\dfrac{1}{r_0}-h.f(r_{i-1,i})}\right) - arc\cos\left(\frac{\dfrac{1}{r_{i-1}}-h.f(r_{i-1,i})}{\dfrac{1}{r_0}-h.f(r_{i-1,i})}\right)\right\} \quad (19)$$

Expression (19) is an approximate value of (18). The value of precession P in one complete cycle, for equal (it is easy to show this) and cumulative movements, is that exceeding $2.\pi$: $P = 2.\theta - 2.\pi$

On the other hand, let's take the expression (17), and do the following action, in order to do some considerations:

$$d\theta = \frac{-du}{\sqrt{(u_0^2 - u^2) - 2.h.(u_0 - u).f(u)}} = \frac{-du}{\sqrt{\left[(u_0^2 - u^2) - 2.h.(u_0 - u)\right]\dfrac{\left[(u_0^2 - u^2) - 2.h.(u_0 - u).f(u)\right]}{\left[(u_0^2 - u^2) - 2.h.(u_0 - u)\right]}}} = \quad (20)$$

$$= \frac{A.du}{\sqrt{\left[(u_0^2 - u^2) - 2.h.(u_0 - u)\right]}} + \frac{B.du}{\sqrt{\dfrac{\left[(u_0^2 - u^2) - 2.h.(u_0 - u).f(u)\right]}{\left[(u_0^2 - u^2) - 2.h.(u_0 - u)\right]}}} \quad (21)$$

In this way, we have separated in two terms the expression in equation (20) and only we need to calculate the constant factors A and B. By simplifying the second denominator, we arrive at the following relation:

$$A.\sqrt{\frac{(u_0+u)-2.h.f(u)}{(u_0+u)-2.h}}+B.\sqrt{\left[(u_0^2-u^2)-2.h.(u_0-u)\right]}=-1 \qquad (22)$$

For $u=u_0, \quad f(u_0)=1 \quad \Rightarrow \quad A=-1$

For $f(u)=\dfrac{u_0+u}{2.h} \quad \Rightarrow \quad B=\dfrac{-1}{\sqrt{(u_0^2-u_f^2)-2.h.(u_0-u_f)}}$ (23)

In obtaining u_f, is only necessary be patient to find this constant value. You can skip the derivation of u_f**.** What are important, really, are the comments to do about the development of equation (22).

$$f(u)=\sqrt{\left(\frac{G.M}{c^2}.(u_0-u)\right)^2+1}-\frac{G.M}{c^2}.(u_0-u)=\frac{u_0+u}{2.h} \;\Rightarrow\; \sqrt{\left(\frac{G.M}{c^2}.(u_0-u)\right)^2+1}=\frac{G.M}{c^2}.(u_0-u)+\frac{u_0+u}{2.h}$$

$$\left(\frac{G.M}{c^2}.(u_0-u)\right)^2+1=\left(\frac{G.M}{c^2}.(u_0-u)\right)^2+2.\frac{G.M}{c^2}.(u_0-u).\frac{u_0+u}{2.h}+\left(\frac{u_0+u}{2.h}\right)^2$$

$$1=2.\frac{G.M}{c^2}.(u_0-u).\frac{u_0+u}{2.h}+\left(\frac{u_0+u}{2.h}\right)^2=\frac{G.M}{c^2.h}u_0^2-\frac{G.M}{c^2.h}u^2+\frac{u_0^2+2.u_0u+u^2}{4.h^2}$$

$$u^2.\left(\frac{1}{4.h^2}-\frac{G.M}{c^2.h}\right)+\frac{u_0}{2.h^2}.u+\left[\frac{G.M}{c^2.h}u_0^2+\frac{u_0^2}{4.h^2}-1\right]=u^2.\left(\frac{1}{4.h^2}-\frac{G.M}{c^2.h}\right)+\frac{u_0}{2.h^2}.u+\left[\frac{u_0^2}{4.h^2}\right]=0$$

$$\Rightarrow\; u_f=\frac{-\frac{u_0}{2.h^2}\pm\sqrt{\left(\frac{u_0}{2.h^2}\right)^2-4..\left(\frac{1}{4.h^2}-\frac{G.M}{c^2.h}\right)\left[\frac{u_0^2}{4.h^2}\right]}}{2.\left(\frac{1}{4.h^2}-\frac{G.M}{c^2.h}\right)}=\frac{-\frac{u_0}{2.h^2}\pm\sqrt{\left[\frac{1}{h^2}\right]}}{2.\left(\frac{1}{4.h^2}-\frac{1}{u_0^2}\right)}=\frac{-\frac{u_0}{4.h^2}\pm\frac{1}{2.h}}{\frac{1}{4.h^2}-\frac{1}{u_0^2}}$$

$$\Rightarrow\; u_f=\frac{-\frac{u_0}{4.h^2}\pm\frac{1}{2.h}}{\frac{1}{4.h^2}-\frac{1}{u_0^2}}$$

$$\Rightarrow\; u_f=\frac{-\frac{u_0}{4.h^2}\pm\frac{1}{2.h}}{\frac{1}{4.h^2}-\frac{1}{u_0^2}}= \qquad \Rightarrow \qquad \boxed{u_f=\frac{-u_0\pm 2.h}{1-\frac{4.h^2}{u_0^2}}} \qquad (24)$$

For $h>u_0$ it is necessary that $\dfrac{G.M}{c^2}>r_0$ and we have to take the negative sign, in order to have radius u_f a positive value. For $h<u_0$, $\dfrac{G.M}{c^2}<r_0$ photon takes a parabolic path, namely a bending

occurs. So, for elliptical paths negative sign holds and for parabolic paths it may be necessary to obtain another condition. Let's leave this so for now, and allow continuing with the elliptical case.

With the calculation of constant factors A and B, we have:

$$d\theta = \frac{-du}{\sqrt{\left[\left(u_0^2 - u^2\right) - 2.h.\left(u_0 - u\right)\right]}} + \frac{B.du}{\sqrt{\frac{\left(u_0 + u\right) - 2.h.f(u)}{\left(u_0 + u\right) - 2.h}}} \tag{25}$$

The first part of the this differential equation has a known solution and it gives us the information that when photon completes a revolution, namely when radius reaches again the value r_0, this integral accomplishes the value of 2π. The second part of the differential equation gives us precisely the value of precession, namely, the additional angle to that of 2π, product of the variation of photon mass.

In this way, we have achieved the existence of precession by only considering a variable mass in the analysis. Although we have given a way to obtain an approximate value of the complete integral by using equation (26), It is welcome any help for exactly calculating such integral.

IV. PLANETS PRECESSION.

By following the same procedure with Photon we resume the obtained relationships:

$$d\theta = \frac{K}{\sqrt{\frac{K^2}{p^2}.\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) - 2.G.M.\left(\frac{1}{r_0} - \frac{1}{r^{\cdot}}\right).\frac{m_0}{m}.r^2.m}}.dr = \frac{dr}{\sqrt{\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) - \frac{2.G.M}{K^2}m_0.m.\left(\frac{1}{r_0} - \frac{1}{r^{\cdot}}\right).r^2}} \tag{26}$$

By doing: $u = \frac{1}{r} \Rightarrow du = -\frac{dr}{r^2};\ h = \frac{G.M.m_0^2}{K^2};\ m = m_0.\left[\sqrt{\left(\frac{G.M}{v^2}.\left(\frac{1}{r_0} - \frac{1}{r}\right)\right)^2 + \frac{V_0^2}{v^2}} - \frac{G.M}{v^2}.\left(\frac{1}{r_0} - \frac{1}{r}\right)\right];$ (27)

And for $m = m_0.f\left(1/r\right) \Rightarrow m = m_0.f(u)$, it can take the following expressions:

$$d\theta = \frac{-du}{\sqrt{.\left(u_0^2 - u^2\right) - 2.h.\left(u_0 - u\right).f(u)}} \tag{28}$$

For $\boxed{f(u) = \sqrt{\left(\frac{G.M}{v^2}.\left(u_0 - u\right)\right)^2 + \frac{V_0^2}{v^2}} - \frac{G.M}{v^2}.\left(u_0 - u\right)}$; and $\boxed{v^2 = V_0^2.\frac{m_0^2}{m^2} - 2.G.M.\left(\frac{1}{r_0} - \frac{1}{r^{\cdot}}\right).\frac{m_0}{m}}$

$$d\theta = \frac{-du}{\sqrt{.(u_0 - h.f(u))^2 - (u - h.f(u))^2}} \quad (29)$$

But, the expressions (28) and (29) do not have any known solution, but as before, it is possible to work on them to demonstrate that in a complete “oval” revolution, after taking again radius r_0, preserving the constant angular momentum law, swept angle is different and grater than 2π.

In here, we approach the swept angle for planets, where $v << c$ by taking $m \cong m_0$ and using the approximate formula used for photons:

$$\theta = \int_{r_0}^{r_a} \frac{-du}{\sqrt{.(u_0 - h.f(u))^2 - (u - h.f(u))^2}} = \sum_{i=1}^{N} \left\{ arc\cos\left(\frac{\frac{1}{r_i} - h.f(r_{i-1,i})}{\frac{1}{r_0} - h.f(r_{i-1,i})} \right) - arc\cos\left(\frac{\frac{1}{r_{i-1}} - h.f(r_{i-1,i})}{\frac{1}{r_0} - h.f(r_{i-1,i})} \right) \right\} \quad (30)$$

Let’s take the differential expression (28), and do the same action done in photon’s analysis:

$$d\theta = \frac{-du}{\sqrt{(u_0^2 - u^2) - 2.h.(u_0 - u).f(u)}} = \frac{-du}{\sqrt{\left[(u_0^2 - u^2) - 2.h.(u_0 - u)\right]\frac{\left[(u_0^2 - u^2) - 2.h.(u_0 - u).f(u)\right]}{\left[(u_0^2 - u^2) - 2.h.(u_0 - u)\right]}}}$$

$$= \frac{-du}{\sqrt{(u_0^2 - u^2) - 2.h.(u_0 - u).f(u)}} = \frac{A.du}{\sqrt{\left[(u_0^2 - u^2) - 2.h.(u_0 - u)\right]}} + \frac{B.du}{\sqrt{\frac{\left[(u_0^2 - u^2) - 2.h.(u_0 - u).f(u)\right]}{\left[(u_0^2 - u^2) - 2.h.(u_0 - u)\right]}}} \quad (31)$$

In this way we have separated in two terms and we only have to calculate the factors A and B, in where the first term has the known solution used for constant masses. Simplifying the expression inside the square root in second denominator, and we have:

$$A.\sqrt{\frac{(u_0 + u) - 2.h.f(u)}{(u_0 + u) - 2.h}} + B.\sqrt{\left[(u_0^2 - u^2) - 2.h.(u_0 - u)\right]} = -1$$

For $u = u_0, \quad f(u_0) = 1 \quad \Rightarrow \quad A = -1$

And for $f(u) = \frac{u_0 + u}{2.h} \quad \Rightarrow \quad B = \frac{-1}{\sqrt{(u_0^2 - u_f^2) - 2.h.(u_0 - u_f)}}$ (32)

V. CONCLUSION

As it can be observed, same analysis done for photon holds for Planets. Although, we were not successful in obtaining **the exact solutions for angle and radius**, in the given treatment to the differential equations, in our opinion, we could accomplish the task of detecting precession inside the equation of motion of photons and planets, and give an approximate calculation of angle. In author’s opinion, it fulfils our expectations. This means also that it is open for further research finding best estimates of angle or its exact solution.

REFERENCES

[1] J A Franco R *Energy in Vectorial Relativity, E≈ m.c²*. Published by JVR on November 16th 2006. JVR **1** (2006) **1** 1-7.

[2] J A Franco R. *Gravitation in Vectorial Relativity: A Force.* Published by JVR on November 16th 2006. JVR **1** (2006) **1** 1-7.

[3] J G Quintero D and J A Franco R. *Gravitational Forces in Vectorial Relativity*. Published by JVR on March 16th 2007. JVR **2** (2007) **1** 33-42.

ANNEX

The following graph indicates the basis for estimating value of angle. We take the mean value as a constant along each interval of integration.