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# Gravitation in Vectorial Relativity

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***ABSTRACT:*** It is known that Kepler's Laws can be derived from the Newton's Law of Universal Gravitation. For doing this, mass is considered as an invariable parameter. Although this consideration works wonderfully to solve most of problems in astronomy calculations, as in all physics, when body's speeds are so high and very precise measurements are required, the referred Kepler Laws do not cope enough what is expected. That's why the General Theory of Relativity materialized. As it was indirectly pointed out by Einstein in 1905, Newton and Kepler Laws do not consider the relativistic variation of mass with its velocity. In the current work it is presented a development that concludes with a modification of the Newton's Universal Force of Gravitation in order to adapt its conception to mass-variation modern view. This development starts with the consideration of a generic variable mass, without establishing any cause of its variation. Later, Newton's principle of attraction of masses was indirectly used for arriving at an expression of the Universal Gravitation Law consistent from the relativistic point of view. It is worth mentioning that this approach was not worked under the Einstein's General Theory of Relativity, but under a three-dimensional environment. There are some surprising results, in addition to the new presentation of the Law of Universal Gravitation, i.e.: the obtained expression for the radius of a circular path of a photon around a massive body appeared to be half of the known Schwarzschild radius..

***KEYWORDS:*** Universal Gravitation Law, Kepler Laws, Angular Momentum, Vectorial Relativity.

## *I INTRODUCTION*

As it is known, photon of light moves rectilinear through an empty space at velocity $c$ and has a mass $m$ that varies with energy, given correctly by the Einstenian expression $m = \frac{E}{c^2}$ [1] [2]. However, energy of a photon can also be characterized by its frequency $\gamma$, through Planck's constant $h$, as $E = h.\gamma$, such that photon's mass is also directly related to its frequency, $m = \frac{h}{c^2}.\gamma$. From here we easily observe that when energy is zero, say when photon is at rest, its frequency and therefore its mass have also a null value.

This work studies the case of a photon, which modifies its path because of the gravitational attraction exerted on its mass by a massive body, when falls in such field. When photon deviates its path, it varies its energy, and therefore its mass. Thus, this is a very good example of variable, or relativistic, mass. This work starts its development about the effect of Gravitation on bodies referred to the photon, which is a particular example of mass, instead of a generic mass, because it supposes an easier work. Later, this study will be extended to any mass. Photon, as it was shown and is well known, has a null rest mass.

In the first part, it is shown that the analysis of the mass of a photon, treated as a variable mass (or relativistic), attracted by a massive body with a gravitational force given by the Newton's Universal

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Gravitation Law, leads to an erroneous and fallacious result. This could indicate that the treatment of gravitation when variable masses are taken into account either is not so simple as when it was done with constant masses, or something is lacking in the Newton's Universal Gravitation Law. This last one is the reason for which in this work is developed a general test of the validity of Newton's Law of Universal Gravitation in any situation, in order to detect its lack of consistency and its correction.

## *II IS THE UNIVERSAL GRAVITATION LAW A GENERAL LAW?*

Although photon does not have rest mass, when it is traveling at speed $c$, it has a non-zero mass given by the known relationship, $m=\frac{E}{c^2}$. Given this feature of photons, they can be attracted by the gravitational field of a massive body and take a curvilinear path as any other body. Let's recall that when a photon changes its mass only changes its frequency, so, its velocity magnitude, $c$, remains constant.

Let's start posing the problem in the following way: Let a photon be attracted by a massive body, approaching onto it at a minimum distance denoted by a radio-vector $R_0$, measured from center of mass of the massive body to a point $P_0$, located at this minimum distance, such that radio-vector $R_0$, forms an angle of 90 degrees with the photon's vector velocity $\mathbf{c}$, at $P_0$. Let's try to find the deflection of the photon, produced onto its variable mass by the gravitational field of the massive body.

By considering gravitational force as central, given by Newton's Universal Law and putting photon mass $m$ as the quotient between the linear Momentum $p$, and the speed of light $c$, $m=\frac{p}{c}$, we can write that:

$$\frac{d\mathbf{p}}{dt}=-F.\mathbf{U}_r=-\frac{G.M.p}{c.r^2}.\mathbf{U}_r \tag{1}$$

where, $G$ is the gravitational constant, $M$ the massive body's mass, and $\mathbf{U}_r$ a unit vector on the direction pointing towards gravitational central force. Minus sign indicates the contrary sense of centrifugal force, $\frac{d\mathbf{p}}{dt}$. Thus, by making: $\mathbf{p}=\left(\frac{p}{c}\right).\mathbf{c}$:

$$\frac{d\mathbf{p}}{dt}=\frac{d\left(\frac{p}{c}.\mathbf{c}\right)}{dt}=\left(\frac{p}{c}.\frac{d(\mathbf{c})}{dt}+\frac{\mathbf{c}}{c}.\frac{dp}{dt}\right)=-\frac{G.M.p}{c.r^2}.\mathbf{U}_\mathbf{r} \tag{2}$$

Expressing in polar form for plane curvilinear motion the vector velocity of light, $\mathbf{c}$, and its acceleration vector, $\frac{d\mathbf{c}}{dt}$, as function of the unit vectors $\mathbf{U}_r$ ($\mathbf{U}_r=\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}$) and $\mathbf{U}_\theta$ ($\mathbf{U}_\theta=-\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j}$). The angle $\theta$, swept by radio-vector $\mathbf{r}$ of the photon's mass in its movement with origin in the center of mass of the massive body, beginning at $P_0$ for $\theta = t = 0$, until a generic point of the trajectory, $P$. Operating on $\mathbf{r} = r.\mathbf{U}_r$, we obtain:

$$\frac{d\mathbf{r}}{dt} = \mathbf{c} = \frac{dr}{dt}.\mathbf{U}_r + r.\frac{d\theta}{dt}.\mathbf{U}_\theta = \frac{dr}{dt}.\mathbf{U}_r + r.\omega.\mathbf{U}_\theta$$

$$\mathbf{c} = \frac{dr}{dt}.\mathbf{U}_r + r.\omega.\mathbf{U}_\theta \tag{3}$$

Also, we can get:

$$\frac{d\mathbf{c}}{dt} = \left[\frac{d^2r}{dt^2} - r.\left(\frac{d\theta}{dt}\right)^2\right].\mathbf{U}_r + \left[r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}.\frac{d\theta}{dt}\right].\mathbf{U}_\theta \tag{4}$$

Substituting these results in (2), dividing by $\frac{p}{c}$, and simplifying, it follows that:

$$\left[\frac{d^2r}{dt^2} - r.\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{p}.\frac{dp}{dt}.\frac{dr}{dt}\right].\mathbf{U}_r + \left[r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}.\frac{d\theta}{dt} + \frac{r}{p}.\frac{dp}{dt}.\frac{d\theta}{dt}\right].\mathbf{U}_\theta = -\frac{G.M}{r^2}.\mathbf{U}_r \tag{5}$$

this vectorial equation originates the following two conditions:

a) What is multiplied by $\mathbf{U}_\theta$, must be zero because gravitational force only have component on $\mathbf{U}_r$, and

b) What is multiplied by $\mathbf{U}_r$ equals $-\frac{G.M}{r^2}$.

From the first condition, we obtain:

$$.\left(r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}.\frac{d\theta}{dt} + \frac{r}{p}.\frac{dp}{dt}.\frac{d\theta}{dt}\right) = 0 \quad \Rightarrow \quad \frac{d^2\theta}{dt^2} = -\left(\frac{2}{r}.\frac{dr}{dt}.\frac{d\theta}{dt} + \frac{1}{p}.\frac{dp}{dt}.\frac{d\theta}{dt}\right)$$

$$\frac{\frac{d^2\theta}{dt^2}}{\frac{d\theta}{dt}} = -\left(\frac{2}{r}.\frac{dr}{dt} + \frac{1}{p}.\frac{dp}{dt}\right) \Rightarrow \text{for, } \omega = \frac{d\theta}{dt} \Rightarrow \frac{\frac{d\omega}{dt}}{\omega} = -\left(2.\frac{dr}{r} + \frac{dp}{p}\right).\frac{1}{dt} \Rightarrow \frac{d\omega}{\omega} = -\left(2.\frac{dr}{r} + \frac{dp}{p}\right)$$

By integrating this last expression from $(\omega_0, r_0, p_0)$ to $(\omega, r, p)$, between $P_0$ and another generic point $P$ of the photon trajectory, it becomes:

$$\Rightarrow \mathbf{ln}\frac{\omega}{\omega_\mathbf{0}} = -2.\mathbf{ln}\frac{r}{r_\mathbf{0}} - \mathbf{ln}\frac{p}{p_\mathbf{0}} \qquad \Rightarrow \qquad \omega.r.p = \omega_\mathbf{0}.r_0^2.p_\mathbf{0} = \text{Constant} \tag{6}$$

If we divide both members by $c$, the speed of light, we recognize in equation (6) another version of the angular momentum for light, constant, as it was expected for maintaining the conservation of Angular Momentum Law. Thus, we have obtained a consistent relationship (6) working on the first condition.

Applying the second condition and substituting $\frac{d\theta}{dt}$ by $\omega$, we have:

$$\frac{d^2r}{dt^2} - r.\omega^2 + \frac{1}{p}.\frac{dp}{dt}.\frac{dr}{dt} = -\frac{G.M}{r^2} \qquad (7)$$

Before working on this expression, let's try to obtain an equivalent one to $\frac{1}{p}.\frac{dp}{dt}$, through the following general Energy definition and known relations (1) and (2):

$$K = \frac{d\mathbf{p}}{dt}.d\mathbf{r} = -G.\frac{M.p}{c.r^2}.\mathbf{U}_r.d\mathbf{r} \;\Rightarrow\; d\mathbf{p}.\frac{d\mathbf{r}}{dt} = d\mathbf{p.c} = dp.c = -G.\frac{M.p}{c.r^2}.\mathbf{U}_r.\left(dr.\mathbf{U}_r + \omega.r.\mathbf{U}_\theta\right)$$

$$dp.c = -G.\frac{M.p}{c.r^2}.dr \qquad \Rightarrow \qquad \frac{dp}{p.dt} = -G.\frac{M}{c^2.r^2}.\frac{dr}{dt} \qquad (8)$$

On the other hand, light speed can be put as: $c^2 = \left(\frac{dr}{dt}\right)^2 + (\omega.r)^2 = q^2 + (\omega.r)^2$, for $q = \frac{dr}{dt}$. (9)

By introducing equations (8) and (9) in (7), we obtain:

$$\frac{d^2r}{dt^2} - r.\omega^2 - \frac{G.M}{c^2r^2}\left(c^2 - r^2.\omega^2\right) = -\frac{G.M}{r^2}.\omega^2$$

Substituting by $\omega = \frac{d\theta}{dt}$: $\frac{d^2r}{dt^2} - r.\omega^2 = -\frac{G.M}{c^2}.\omega^2 \quad \Rightarrow \quad \frac{d^2r}{d\theta^2} - r = -\frac{G.M}{c^2}$ (10)

This second order differential equation, inhomogeneous, has as general solution:

$$r = A.e^\theta + B.e^{-\theta} - \frac{G.M}{2.c^2}\left[e^\theta + e^{-\theta} - 2\right]$$

Where initial conditions for obtaining the values of constants A and B, are: for angle $\theta = 0$, $r = R_0$, and $\frac{dr}{d\theta} = 0$, because radius is minimum at $P_0$. Thus, the explicit and general solution for any generic point $P$, on the trajectory, becomes:

$$r = \frac{R_0}{2}.e^\theta + \frac{R_0}{2}.e^{-\theta} - \frac{G.M}{2.c^2}\left[e^\theta + e^{-\theta} - 2\right] = R_0.\cosh\theta - \frac{G.M}{c^2}\left[\cosh\theta - 1\right] \qquad (11)$$

Let's analyze this result. If instead of a massive body a mathematical point existed single ($M = 0$) "attracting" the photon, instead of a curvilinear path the photon will describe a rectilinear movement,

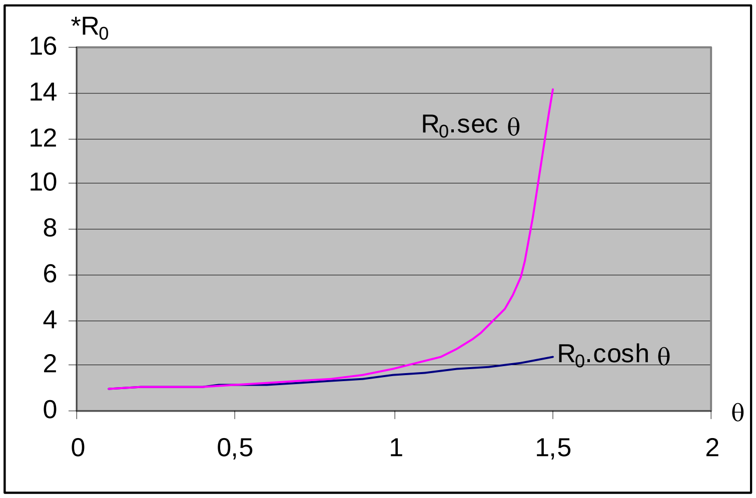


Fig. 1. Differences between classical equation (12) and equation (11)

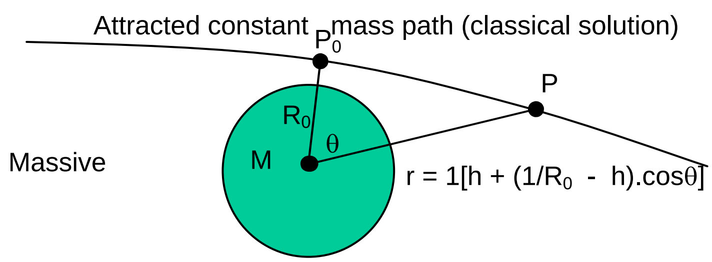


Fig. 2 Classical solution

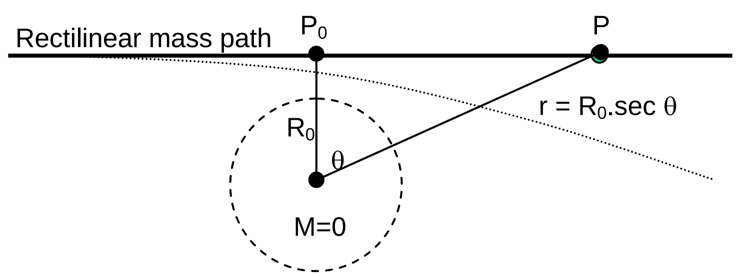


Fig. 3 Solution of equation (12). Classical solution for M = 0

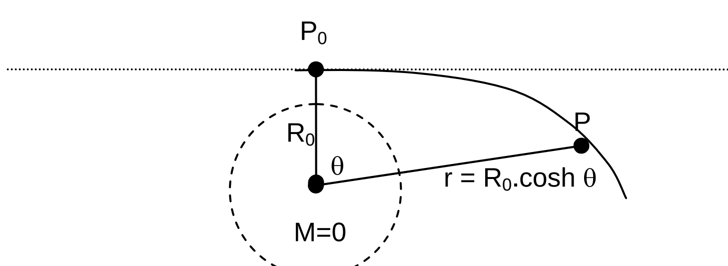


Fig. 4 Solution of equation (11) for M = 0 (!).

and its mass, energy and frequency will remain constants. In such situation the well-known first Kepler's law applies and also its general solution:

$$r = \frac{1}{h + \left(\frac{1}{R_0} - h\right).\cos\theta}; \quad \text{where} \quad h = \frac{G.M}{c^2.R_0^2}; \text{ for } M = 0 \ \Rightarrow \ r = R_0.\sec\theta \qquad (12)$$

But, the solution given by equation (11) is $r = R_0.\cosh\theta$.

Then, equation (11), for $M = 0$, for any $\theta$, gives as result for radius: $r = R_0.\cosh\theta$, which is inconsistent and not valid. In fact, its solution differs from the correct one given by equation (12).

Thus, we have to do an inventory of present assumptions in this development:

1) Speed of light is constant as in rectilinear as in curvilinear motion. It is a logical assumption. It will be maintained.
2) Definition of photon mass, variable. It comes from the previous assumption.
3) Gravitational forces are central. It has worked for constant moving masses. We will sustain it.
4) Definition of Newton's Gravitational Force has been accepted as valid for movable variable masses. In Author's opinion, as it has been shown in previous development, it can be presumed to be valid only between static attracting bodies.

Next, let's look for a gravitational law expression, so that logical and expected results are obtained.

### *III DERIVATION OF THE GRAVITATIONAL LAW FOR PHOTONS*

Let's try to find out where the weakness of Newton's Gravitational Force could be.

Thus, let's restart the analysis of the photon motion under the effect of a gravitational force of a massive body. Let's work with general equations trying to avoid as possible, assumptions. But, let's suppose, by now, that we don't know the expression of the Newton's universal gravitational force. The second condition in equation (7) keeping as central the unknown gravitational force, $F$ :

$$\frac{d^2r}{dt^2} - r.\omega^2 + \frac{1}{p}.\frac{dp}{dt}.\frac{dr}{dt} = -\frac{F.c}{p} \qquad (13)$$

And preserving the definition of the gravitational field:

$$\frac{d\mathbf{p}}{dt} = -F.\mathbf{U_r} \quad \Rightarrow \quad \frac{d\mathbf{p}.d\mathbf{r}}{dt} = -F.d\mathbf{r}.\mathbf{U_r} \quad \Rightarrow \quad dp.c = -F.dr$$

$$\frac{dp}{p} = -\frac{F}{p.c}.dr = -\frac{\mathrm{G}}{c^2}.dr \quad \text{for, } F = \frac{\mathrm{G}.p}{c}; \quad \text{or, } \mathrm{G} = \frac{F.c}{p} \qquad (14)$$

Where G , denotes the gravitational field. From (9) we can put (13) as:

$$\frac{d^2r}{dt^2}-r.\omega^2-\frac{F}{p.c}.\frac{dr}{dt}.\frac{dr}{dt}=\frac{d^2r}{dt^2}-r.\omega^2-\frac{\mathrm{G}}{c^2}.\left(c^2-\omega^2.r^2\right)=-\mathrm{G}$$

We will try to obtain a general relationship, for an unknown gravitational field G :
:

$$\frac{d^2r}{dt^2}-r.\omega^2+\frac{\mathrm{G}}{c^2}\omega^2.r^2=0 \qquad (15)$$

Let's obtain other general relationships. We know, by the law of conservation of the angular momentum, that:

$$\frac{dp}{p}=-\frac{d(\omega.r^2)}{\omega.r^2}$$

So, from (14) we can establish the following relation:

$$\frac{d(\omega.r^2)}{\omega.r^2}=\frac{\mathrm{G}}{c^2}.dr \qquad \Rightarrow \qquad \omega.r^2.d(\omega.r^2)=\left(\omega.r^2\right)^2\frac{\mathrm{G}}{c^2}.dr$$

Making the integration from the minimum radius (perihelion) to a generic radius:

$$\omega^2.r^4-\omega_0^2.r_0^4=2.\int_{r_0}^{r}\left(\omega.r^2\right)^2\frac{\mathrm{G}}{c^2}.dr$$

What is just like to write:

$$\frac{1}{p^2}-\frac{1}{p_0^2}=\frac{2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2\frac{\mathrm{G}}{c^2}.dr \qquad \Rightarrow \qquad p^2=\frac{p_0^2}{1+\frac{2.p_0^2}{c^2.K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr}$$

From here we can obtain

$$p^2=\frac{p_0^2}{1+\frac{2.p_0^2}{c^2.K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr}=\frac{c^2.p_0^2}{c^2+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr}=\frac{\omega_0^2.r_0^2.p_0^2}{c^2+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr}$$

$$p^2=\frac{\omega_0^2.r_0^2.p_0^2}{c^2+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr}=\frac{\omega_0^2.r_0^4.p_0^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)+\frac{\omega_0^2.r_0^4.p_0^2}{r^2}}{c^2+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr}$$

$$p^2.\left[c^2-\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr\right]=\omega_0^2.r_0^4.p_0^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)+\frac{\omega_0^2.r_0^4.p_0^2}{r^2}$$

$$p^2.\left[c^2-\frac{\omega_0^2.r_0^4.p_0^2}{p.^2r^2}+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr\right]=\omega_0^2.r_0^4.p_0^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)$$

$$p^2.\left[c^2-\omega^2.r^2+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr\right]=\omega_0^2.r_0^4.p_0^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)$$

$$p^2.\left[q^2+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr\right]=\omega_0^2.r_0^4.p_0^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)$$

By remembering that $c^2=q^2+\omega^2.r^2$, this takes us to a relation similar to that of the solution for the classical case of constant mass:

$$\Rightarrow \qquad q^2=\frac{\omega_0^2.r_0^4.p_0^2}{p^2}.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)-\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr$$

On the other hand,

$$\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr=\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\frac{K^2}{p^2}.(-c^2\frac{dp}{p})=-p_0^2.c^2.\left(\frac{1}{p_0^2}-\frac{1}{p^2}\right)$$

In this way, we can write down the following **exact** expression:

$$q^2=\omega^2.r^4.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)+p_0^2.c^2.\left(\frac{1}{p_0^2}-\frac{1}{p^2}\right) \tag{16}$$

A similar expression is found in classical analysis for constant mass (except for the second term at right):

$$q^2=v_0^2.r_0^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)-2.G.M.\left(\frac{1}{r_0}-\frac{1}{r}\right) \tag{17}$$

Let's work on this expression to establish some criteria to orientate our research. When in elliptic motion a constant mass, coming from perihelion, $r_0$, pass through aphelion, $q$ becomes null. That is what the classical equation (17) means. Evaluating (17) at aphelion we obtain:

$$v_0^2.r_0^2 = \frac{2.G.M}{\frac{1}{r_0} + \frac{1}{r_A}}$$

Changing the roles of parameters or which is just like to say that we are evaluating (17) at perihelion but coming from aphelion. Thus, we obtain:

$$v_A^2.r_A^2 = \frac{2.G.M}{\frac{1}{r_A} + \frac{1}{r_0}}$$

From these two results, it follows that for any constant mass that moves elliptically around a mass $M$ : $v_0^2.r_0^2 = v_A^2.r_A^2 \;\Rightarrow\; \omega_0^2.r_0^4 = \omega_A^2.r_A^4$

Observe that this last expression is the angular momentum for constant mass, evaluated at aphelion and perihelion: $m^2.\omega_0^2.r_0^4 = m^2\omega_A^2.r_A^4$. By taking this classical solution as a clue or as a starting point for developing our study in a similar manner for photon's motion, and remembering from the result obtained by applying the first condition for photon, $\omega_0.r_0^2.p_0 = \omega.r^2.p$, allow the second term at the right in equation (16) being modified as:

$$q^2 = \omega^2.r^4.\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) - 2.G.M.\left(\frac{1}{r_0} - \frac{1}{r}\right) \tag{18}$$

In this way, we would be in the same situation as before, when we were in the classical case. For an elliptic motion of the variable mass of **photon**, coming from perihelion, $r_0$, and just passing through aphelion, the radial velocity $q$ becomes null at this moment. That is what is deduced from equation (18). Evaluating (18) at aphelion and at perihelion, as before but now for photon, we obtain:

$$\omega_A^2.r_A^4 = c^2.r_A^2 = \frac{2.G.M}{\frac{1}{r_0} + \frac{1}{r_A}} \qquad \omega_0^2.r_0^4 = c^2.r_0^2 = \frac{2.G.M}{\frac{1}{r_A} + \frac{1}{r_0}} \quad \Rightarrow \quad c^2.r_A^2 = c^2.r_0^2 \quad \Rightarrow \quad r_A = r_0$$

As we see, these two equations lead to the contradiction that radiuses of perihelion and aphelion are forced to be equal given the constant speed of photon, namely, only circular motion is allowed by equation (18) in this case. How to break this limitation? From analyzing what happened with classical case, we conclude that our current goal will be to obtain a factor inside equation (18) that solves the inconsistency of radiuses, preserving the constancy of angular momentum for photon's motion. Then, we have to look for an expression slightly different to that of (18) such that after evaluating, radiuses contradiction becomes broken and we can arrive at the equality of angular momentum, at perihelion and aphelion, i.e.:

$$c^2.r_0^2.p_0^2 = c^2.r_A^2.p_A^2$$

We found that the following general expression for $q^2$, with the factor of the linear momentums quotient multiplying the second term of the right, fulfills the requirement of the angular momentum's conservation and radiuses' limitation is broken:

$$q^2 = \frac{\omega_0^2 . r_0^4 . p_0^2}{p^2} . \left( \frac{1}{r_0^2} - \frac{1}{r^2} \right) - 2.G.M . \frac{p_0}{p} . \left( \frac{1}{r_0} - \frac{1}{r} \right) \tag{19}$$

Previous one is the general expression starting from perihelion. By considering as starting from aphelion, the following general expression shows up:

$$q^2 = \frac{\omega_A^2 . r_A^4 . p_A^2}{p^2} . \left( \frac{1}{r_A^2} - \frac{1}{r^2} \right) - 2.G.M . \frac{p_A}{p} . \left( \frac{1}{r_A} - \frac{1}{r} \right) \tag{20}$$

Evaluating equations (19) at aphelion and (20) at perihelion, we obtain the expected expression of constant angular momentum, which was our goal:

$$c^2 . r_0^2 . \frac{p_0}{p_A} = c^2 . r_A^2 . \frac{p_A}{p_0} = \frac{2.G.M}{\dfrac{1}{r_0} + \dfrac{1}{r_A}} \quad \Rightarrow \quad c^2 . r_0^2 . p_0^2 = c^2 . r_A^2 . p_A^2 = p_0 . p_A . \frac{2.G.M}{\dfrac{1}{r_0} + \dfrac{1}{r_A}}$$

For circular motion of photon around the massive body where, $r_A = r_0; \quad p_A = p_0$, and the following result arises:

$$c^2 = \frac{G.M}{r_0} \qquad \Rightarrow \qquad r_0 = \frac{G.M}{c^2}$$

Observe carefully this probable exact result, according to this work. The circular motion of a photon of light depends on the mass $M$ of the massive body, and has a radius that is a unique $r_0$. For instance, if the boundary of the body, at $r = r_B$, is inside the radius of the circular motion of the photon, i.e., $r_B < r_0$, then this body is known as a black hole, because light only can have a concentric-spiral motion within this border. Also, it is worth noting that the obtained result for this radius in this work, $r_0 = G.M / c^2$, is half of the radius known in the relativistic jerk as Schwarzschild radius [3 ].

After this comment, let's continue. Observe that the expression (19) takes implicit assuming the equality given next, which will be continuously checked throughout all this work.

$$p_0^2 . c^2 . \left( \frac{1}{p_0^2} - \frac{1}{p^2} \right) = -2.G.M . \frac{p_0}{p} . \left( \frac{1}{r_0} - \frac{1}{r} \right) \tag{21}$$

Starting from the expression (19), let's try now to obtain the expression for the gravitational field. For that, our strategy will be to take derivatives relative to time, in order to construct an equation similar to the equation (13), where the force and therefore the gravitational field are unknowns. Operating on equation (19):

$$q^2 = \frac{\omega_0^2 . r_0^4 . p_0^2}{p^2} . \left( \frac{1}{r_0^2} - \frac{1}{r^2} \right) - 2.G.M . \frac{p_0}{p} . \left( \frac{1}{r_0} - \frac{1}{r} \right) = \frac{K^2}{p^2} . \left( \frac{1}{r_0^2} - \frac{1}{r^2} \right) - 2.G.M . \frac{p_0}{p} . \left( \frac{1}{r_0} - \frac{1}{r} \right)$$

$$2.q.dq = \frac{2.K^2}{p^2} . \frac{dr}{r^3} - 2. \left( \frac{1}{r_0^2} - \frac{1}{r^{2.}} \right) \frac{K^2}{p^3} .dp + 2.G.M . \left( \frac{1}{r_0} - \frac{1}{r^{\cdot}} \right) . \frac{p_0}{p^2} .dp - 2.G.M . \frac{p_0}{p} . \frac{dr}{r^2}$$

$$q.dq = \frac{K^2}{p^2} . \frac{dr}{r^3} - \left( \frac{1}{r_0^2} - \frac{1}{r^{2.}} \right) \frac{K^2}{p^3} .dp + G.M . \left( \frac{1}{r_0} - \frac{1}{r^{\cdot}} \right) . \frac{p_0}{p^2} .dp - G.M . \frac{p_0}{p} . \frac{dr}{r^2}$$

$$q . \frac{dq}{dr} = \omega^2 .r + \frac{1}{r_0^2} \frac{K^2}{p^2} . \frac{\mathsf{G}}{c^2} - \frac{1}{r^2} \frac{K^2}{p^2} . \frac{\mathsf{G}}{c^2} - G.M . \frac{p_0}{p} . \frac{\mathsf{G}}{c^2} \left( \frac{1}{r_0} - \frac{1}{r} \right) - G.M . \frac{p_0}{p} . \frac{1}{r^2}$$

Thus, replacing and regrouping, by observing that: $q . \frac{dq}{dr} = \frac{dr}{dt} . \frac{d}{dr} \left( \frac{dr}{dt} \right) = \frac{d^2 r}{dt^2}$

$$\frac{d^2 r}{dt^2} = \omega^2 .r - \frac{\omega^2 .r^2}{c^2} . \mathsf{G} + \left[ \mathsf{G} . \frac{p_0^2}{p^2} - G.M . \frac{p_0}{p} . \frac{\mathsf{G}}{c^2} \left( \frac{1}{r_0} - \frac{1}{r} \right) - \frac{G.M}{r^2} . \frac{p_0}{p} \right]$$

Here we must make the term between parenthesis null, because equation (15), for the case of the photon, does not contain this term. Since within the parenthesis it appears the Gravitation Field like variable, this condition takes us to obtain it. In fact, from:

$$\mathsf{G} . \frac{p_0^2}{p^2} - G.M . \frac{p_0}{p} . \frac{\mathsf{G}}{c^2} \left( \frac{1}{r_0} - \frac{1}{r} \right) - \frac{G.M}{r^2} . \frac{p_0}{p} = 0$$

We finally obtain the expression for the gravitation Field:

$$\mathsf{G} = \frac{\frac{G.M}{r^2} . \frac{p_0}{p}}{\frac{p_0^2}{p^2} - \frac{G.M}{c^2} . \frac{p_0}{p} . \left( \frac{1}{r_0} - \frac{1}{r} \right)}$$

If we substitute in this obtained expression, the expression previously assumed by that exact one in equation (21), we obtain a more simplified expression for Gravitational Field:

$$\mathrm{G}=\frac{\frac{G.M}{r^2}.\frac{p_0}{p}}{\frac{p_0^2}{p^2}-\frac{G.M}{c^2}.\frac{p_0}{p}.\left(\frac{1}{r_0}-\frac{1}{r}\right)}=\frac{\frac{G.M}{r^2}.\frac{p_0}{p}}{\frac{p_0^2}{p^2}+\frac{p_0^2}{2}.\left(\frac{1}{p_0^2}-\frac{1}{p^2}\right)}\quad\Rightarrow\quad \mathrm{G}=\frac{\frac{2.G.M}{r^2}}{\frac{p}{p_0}+\frac{p_0}{p}} \tag{22}$$

Observe that, according to this work, we have arrived at an exact expression of the Gravitational field for photons. Also, it can be observed that gravitational field G depends, not only on the radius but on the linear momentum of the photon. This means also that Newton's Gravitational Force for photons will have the following new presentation:

$$F=\frac{\frac{2.G.M.m}{r^2}}{\frac{p}{p_0}+\frac{p_0}{p}}=\frac{\frac{2.G.M.p}{r^2.c}}{\frac{p}{p_0}+\frac{p_0}{p}} \tag{23}$$

A first conclusion of this study is that the original Newton's Law of Universal Gravitation as it is known and applied in the case of photon, from expression (23), according to this development, is valid only for circular motion of photon, where, $r=r_0;\quad p=p_0$.

By using another process of independent derivation, in order to check consistency of the assumption in (21), we should obtain another way to get to the expression of the Field. Repeating equation (21):

$$-2.G.M.\left(\frac{1}{r_0}-\frac{1}{r}\right).\frac{p_0}{p}=p_0^2.c^2.\left(\frac{1}{p_0^2}-\frac{1}{p^2}\right)$$

Simplifying and separating variables, with the advisable multiplications, we obtain:

$$\Rightarrow\quad \frac{-2.G.M}{c^2.r_0}+\frac{2.G.M}{c^2.r}=\left(\frac{p}{p_0}-\frac{p_0}{p}\right)$$

Differentiating both members, multiplying by -1 and replacing:

$$\frac{2.G.M}{c^2.r^2}.dr=-\left(\frac{dp}{p_0}+\frac{p_0.dp}{p^2}\right)=-\frac{dp}{p}.\left(\frac{p}{p_0}+\frac{p_0}{p}\right)=\frac{\mathrm{G}}{c^2}.dr.\left(\frac{p}{p_0}+\frac{p_0}{p}\right)$$

$$\frac{2.G.M}{r^2}=\mathrm{G}.\left(\frac{p}{p_0}+\frac{p_0}{p}\right)\quad\Rightarrow\quad \mathrm{G}=\frac{\frac{2.G.M}{r^2}}{\left(\frac{p}{p_0}+\frac{p_0}{p}\right)}$$

We obtain that both expressions obtained for the Gravitation Field are equal, and therefore the expected consistency in the assumed relation is obtained. This is a valuable result in this search of the expression of the Field, because different processes of differentiation originate linearly independent equations. Yet we will continue checking the expression assumed through other routes.

We will try next to obtain the relationship between radius $r$ and linear momentum $p$. From the assumption (21), we easily obtain the expression for $p$, and also for $p_0$:

$$p = p_0 \cdot \left[ \sqrt{\left( \frac{G.M}{c^2} \cdot \left( \frac{1}{r_0} - \frac{1}{r} \right) \right)^2 + 1} - \frac{G.M}{c^2} \cdot \left( \frac{1}{r_0} - \frac{1}{r} \right) \right]$$
$$p_0 = p \cdot \left[ \sqrt{\left( \frac{G.M}{c^2} \cdot \left( \frac{1}{r_0} - \frac{1}{r} \right) \right)^2 + 1} + \frac{G.M}{c^2} \cdot \left( \frac{1}{r_0} - \frac{1}{r} \right) \right] \qquad (24)$$

Let's make some conceptual checks on These results. We should expect that for $r = \infty$, the linear momentum be a positive value, finite and less than the momentum $p_0$ at the closest point at $r_0$, where photon has its maximum energy:

$$p = p_0 \cdot \left[ \sqrt{\left( \frac{G.M}{c^2} \cdot \left( \frac{1}{r_0} \right) \right)^2 + 1} - \frac{G.M}{c^2} \cdot \left( \frac{1}{r_0} \right) \right] > 0 \qquad \Rightarrow \qquad p < p_0$$

In fact, thus it is, which indicates a coherent result.

Additionally, it is also expected that for an infinite radius $r = \infty$, because angular velocity is null, $\omega = 0$ and according to the general equation for velocities (9), the value of $q$ should be equal to the velocity of light $c$. From the exact expression in (16), obviously, we should have this result. Thus for $r = \infty$, we have:

$$q^2 = \frac{\omega_0^2 . r_0^4 . p_0^2}{p^2} \cdot \left( \frac{1}{r_0^2} - \frac{1}{r^2} \right) + p_0^2 . c^2 \cdot \left( \frac{1}{p_0^2} - \frac{1}{p^2} \right) = \frac{c^2 . p_0^2}{p^2} + c^2 - \frac{c^2 . p_0^2}{p^2} = c^2$$

Thus, as it was expected, we have:

$$q^2 = c^2$$

On the other hand, operating with the assumed expression in (21), by evoking the conservation of the angular momentum and equation of velocities (9) we have:

$$q^2 = \frac{\omega_0^2 . r_0^4 . p_0^2}{p^2} . \left( \frac{1}{r_0^2} - \frac{1}{r^2} \right) - 2.G.M. \frac{p_0}{p} . \left( \frac{1}{r_0} - \frac{1}{r} \right) = \left( \omega^2 . r^2 \right) . r^2 . \left( \frac{1}{r_0^2} - \frac{1}{r^2} \right) - 2.G.M. \frac{p_0}{p} . \left( \frac{1}{r_0} - \frac{1}{r} \right)$$

$$q^2 = \left( c^2 - q^2 \right) . \left( \frac{r^2}{r_0^2} - 1 \right) - 2.G.M. \frac{p_0}{p} . \left( \frac{1}{r_0} - \frac{1}{r} \right)$$

$$q^2 . \left( 1 + \frac{r^2}{r_0^2} - 1 \right) = \frac{r^2}{r_0^2} . q^2 = c^2 . r^2 . \left( \frac{1}{r_0^2} - \frac{1}{r^2} \right) - 2.G.M. \frac{p_0}{p} . \left( \frac{1}{r_0} - \frac{1}{r} \right)$$

$$q^2 = c^2 . r_0^2 . \left( \frac{1}{r_0^2} - \frac{1}{r^2} \right) - 2.G.M. \frac{p_0}{p} . \frac{r_0^2}{r^2} . \left( \frac{1}{r_0} - \frac{1}{r} \right) \xrightarrow{r \to \infty} q^2 = \frac{c^2 . r_0^2}{r_0^2} = c^2$$

Namely, the assumed expression is consistent as in circular motion as in elliptic, or parabolic and respects the conservation of angular momentum. In sum, until now in the checking process the assumption has successfully passed.

## *IV THE GRAVITATIONAL LAW FOR ANY PAIR OF MASSES.*

Let's try to follow a similar procedure for developing gravitational attraction between two generic masses and obtaining a general expression similar to that of equation (19).

The second condition applied to a mass rotating around another one originates an equation similar to that of following equation (13).

$$\frac{d^2 r}{dt^2} - \omega^2 . r + \left( \frac{1}{p} \frac{dp}{dt} - \frac{1}{v} \frac{dv}{dt} \right) \frac{dr}{dt} = -\frac{F}{m} = -\mathrm{G} \tag{25}$$

The exact expression that is obtained for masses, analogous to that of equation (16), by following a similar procedure to that of photon's, developed into:

$$q^2 = \frac{K^2}{m^2} . \left( \frac{1}{r_0^2} - \frac{1}{r^2} \right) + \frac{K^2}{r_0^2} . \left( \frac{1}{m_0^2} - \frac{1}{m^2} \right) - \left( V_0^2 - v^2 \right) \tag{26}$$

And the expression, similar to (20) that ensures conservation of angular momentum became:

$$q^2 = \frac{K^2}{m^2} . \left( \frac{1}{r_0^2} - \frac{1}{r^2} \right) - 2.G.M. \left( \frac{1}{r_0} - \frac{1}{r} \right) . \frac{m_0}{m} \tag{27}$$

By taking derivatives in equation (27) for obtaining the corresponding equation for later equaling to that in (25), is obtained the general expression of the gravitational field:

$$\mathrm{G} = \frac{\dfrac{2.G.M}{r^2}.\dfrac{v}{V_0} - v.\dfrac{dv}{dr}.\left(\dfrac{p_0}{p} - \dfrac{p}{p_0}\right)}{\left(\dfrac{p}{p_0} + \dfrac{p_0}{p}\right)} \tag{28}$$

Observe that this dynamical expression (28) for the gravitational field reduces to that of photon, for $v = c$, given the constancy of the speed of light. Expression (28) for circular movement of a mass $m$ around another mass $M$, where for $r = r_0$, $p = p_0$, $m = m_0$ and $v = V_0$, namely for a constant mass, reduces consistently to the known expression for the Newtonian gravitational field: $\mathrm{G} = \dfrac{G.M}{r^2}$. At this moment we can reach the conclusion that the original Newton Gravitational Law for any masses is valid only for constant masses which is the same as saying that it is valid only for rest masses or at constant speed in circular motion or in rectilinear motion. In this way, a general expression of the gravitational force $F$ between two masses: $m$ and $M$, that corrects that of Newton, finally obtained is:

$$F = \frac{\dfrac{2.G.M}{r^2}.\dfrac{m.v}{V_0} - m.v.\dfrac{dv}{dr}.\left(\dfrac{p_0}{p} - \dfrac{p}{p_0}\right)}{\left(\dfrac{p}{p_0} + \dfrac{p_0}{p}\right)} = \tag{29}$$

Other relationships that can be obtained are:

$$m = m_0.\left[\sqrt{\left(\frac{G.M}{V_0^2}.\left(\frac{1}{r_0} - \frac{1}{r}\right)\right)^2 + \frac{v^2}{V_0^2}} - \frac{G.M}{V_0^2}.\left(\frac{1}{r_0} - \frac{1}{r}\right)\right]$$
$$m_0 = m.\left[\sqrt{\left(\frac{G.M}{v^2}.\left(\frac{1}{r_0} - \frac{1}{r}\right)\right)^2 + \frac{V_0^2}{v^2}} + \frac{G.M}{v^2}.\left(\frac{1}{r_0} - \frac{1}{r}\right)\right] \qquad \text{(Mass)} \tag{30}$$

$$p = p_0.\left[\sqrt{\left(\frac{G.M}{v.V_0}.\left(\frac{1}{r_0} - \frac{1}{r}\right)\right)^2 + 1} - \frac{G.M}{v.V_0}.\left(\frac{1}{r_0} - \frac{1}{r}\right)\right]$$
$$p_0 = p.\left[\sqrt{\left(\frac{G.M}{v.V_0}.\left(\frac{1}{r_0} - \frac{1}{r}\right)\right)^2 + 1} + \frac{G.M}{v.V_0}.\left(\frac{1}{r_0} - \frac{1}{r}\right)\right] \qquad \text{(Momentum)} \tag{31}$$

$$q = \sqrt{\frac{K^2}{m^2}.\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) - 2.G.M.\left(\frac{1}{r_0} - \frac{1}{r^{\cdot}}\right).\frac{m_0}{m}}$$ (Radial velocity) (32)

$$v^2 = V_0^2.\frac{m_0^2}{m^2} - 2.G.M.\left(\frac{1}{r_0} - \frac{1}{r^{\cdot}}\right).\frac{m_0}{m}$$ (Generic velocity) (33)

We should say that the only aspect that we have not considered yet here and that completes this study in the sense of defining relativistic gravitation, is the different presentations of mass appearing in this development, which uses the following mass definitions, measured by an observer at the fixed point $P_0$: the rest mass $\overset{0}{m}$ (change of notation is to distinguish it from the massive mass M) , the initial mass $m_0$, when it is moving at velocity $V_0$ and is passing by $P_0$ at the minimum distance $r_0$ from M; and the generic value of mass $m$, measured when it has another generic velocity $v$ and another generic radius $r$, as so it was obtained in equation (4) in the Review presented in this issue [3]: "Mass in Vectorial Relativity", based on Franco's work [5], where, recalling (with the new notation), they were related among them as:

$$m = \frac{\overset{0}{m}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \qquad m_0 = \frac{\overset{0}{m}}{\left(1 - \frac{V_0^2}{c^2}\right)^{\frac{3}{2}}}$$

## *V CONCLUSIONS*

As in previous definitions of mass or energy, again we are in the same situation: It is necessary to check the experimental validation of this approach. For example, until now we have not found any relation between gravitation and time as it is in Einstein's General Theory of Relativity or between the speed of light and gravitation as it is found in the speed of a radar signal when it goes opposite to or parallel to the gravitational force. The accuracy of new definitions of mass, energy or Gravitational Force rigorously obtained, according to us, in this work and in previous ones, referred in the Reviews published in this issue, will probably require further research and complex experiments with known rest masses accelerated at speeds close to that of light in order to establish the correctness of our work. These tasks probably could be possible to achieve by next years. See in the "News", in the first part of this Journal, the article **"European new particle accelerator: The Large Hadron Collider (LHC)".**

## *REFERENCES*

[1] Albert Einstein. *Zur Elektrodynamik bewegter Körper,* Annalen der Physic 17, 1905, pp. 891-921. English version. *On the Electrodynamics of Moving Bodies*. http://www.fourmilab.ch/etexts/einstein/specrel/www/.

[2] J. A. Franco R. *Energy in Vectorial Relativity, E≈ m.c*$^2$. Published in this issue.

[3] J. Foster and J. D. Nightingale. *A Short Course of General Relativity*, Longman Inc., New York, 1979. Page 123.

[4] J. A. Franco R. *Vectorial Lorentz Transformations.* Published by EJTP on February 25th, 2006. EJTP **9** (2006) 35-64. http://www.ejtp.com/ , http://www.ejtp.net/ .

[5] J G Quintero D and J. A. Franco R. *Mass in Vectorial Relativity.* Published in this issue.